# Time Series Analysis

Solutions to problems in Chapter 7

IMM

# Solution 7.1

*Question 1.*

$$kX_t = (1 + B + B^2 + ...B^{k-1})\epsilon_t$$

As

$$\begin{aligned} \theta(B) &= (1 + B + ... + B^{k-1}) \text{ , and} \\ B\theta(B) &= (B + B^2 + ... + B^k) \text{ ,} \end{aligned}$$

we get by subtraction

$$\begin{aligned} (1 - B)\theta(B) &= (1 - B^k) \Rightarrow \\ \theta(B) &= \frac{1 - B^k}{1 - B} \end{aligned}$$

Thus, the spectral density for $\{X_t\}$ can written as

$$\begin{aligned} f_x(\omega) &= \frac{\sigma_\epsilon^2}{2\pi} \frac{(1 - e^{i\omega k})(1 - e^{-i\omega k})}{k^2(1 - e^{i\omega})(1 - e^{-i\omega})} \\ &= \frac{\sigma_\epsilon^2}{2\pi} \frac{(2 - 2\cos(k\omega))}{k^2(2 - 2\cos(\omega))} \\ &= \frac{\sigma_\epsilon^2}{2\pi} \frac{\sin^2(0.5k\omega)}{k^2 \sin^2(0.5\omega)} \end{aligned}$$

*Question 2.*

An expression for the window specified in frequency domain is found as the fourier transform of the window in the time domain, i.e.

$$\begin{aligned} G(\omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} g(t)e^{-i\omega t} dt = \frac{1}{2\pi} \int_{-T}^{T} \frac{a}{T} e^{-i\omega t} dt \\ &= \frac{-a}{2\pi T i\omega} \left(e^{-i\omega T} - e^{i\omega T}\right) \\ &= \frac{ai}{2\pi T\omega} (cos(\omega T) - i\sin(\omega T) - \cos(\omega T) - i\sin\omega T) \\ &= \frac{a}{\pi} \frac{\sin(\omega T)}{\omega T} \end{aligned}$$

# Solution 7.2

*Question 1.*

$$\begin{aligned}
&\lambda_k \geq 0 \quad k \Rightarrow \\
&1 - 2a + 2a\cos\left(\frac{\pi k}{M}\right) \geq 0 \quad k \Rightarrow \\
&1 - 2a - 2a \geq 0 \Rightarrow \\
&a \leq \underline{\underline{0.25}}
\end{aligned}$$

*Question 2.*

Using the general Tukey window the following spectral estimate is obtained

$$\begin{aligned}
\hat{f}(\omega) &= \frac{1}{2\pi}\left(C_0 + 2\sum_{k=1}^{M}\left(1 - 2a + 2a\cos\left(\frac{\pi k}{M}\right)\right)C_k\cos(\omega k)\right) \\
&= \frac{1}{2\pi}\left(C_0 + 2\sum_{k=1}^{M}(1-2a)C_k\cos(\omega k)\right) \\
&+ \frac{1}{2\pi}\left(\sum_{k=1}^{M}\left(aC_k\cos\left(\omega k + \frac{\pi}{M}k\right) + aC_k\cos\left(\omega k - \frac{\pi}{M}k\right)\right)\right) \\
&= \underline{\underline{a\hat{f}_1\left(\omega - \frac{\pi}{M}\right) + (1-2a)\hat{f}_1 + a\hat{f}_1\left(\omega + \frac{\pi}{M}\right)}}
\end{aligned}$$

*Question 3.*

$$\frac{\nu \hat{f}(\omega)}{f(\omega)} \in \chi^2(\nu) \Rightarrow$$
$$\text{Var}\left[\frac{\nu \hat{f}(\omega)}{f(\omega)}\right] = 2\nu \Rightarrow$$
$$\text{Var}\left[\frac{\hat{f}(\omega)}{f(\omega)}\right] = \frac{2}{\nu}$$

The degrees of freedom are

$$\nu = 2N/\left(\sum_{k=-M}^{M} \lambda_k^2\right) \ ,$$

where

$$\begin{aligned}
\sum_{k=-M}^{M} \lambda_k^2 &= \sum_{k=-M}^{M} \left(1 - 2a - 2a\cos\left(\frac{\pi k}{M}\right)\right)^2 \\
&= \sum_{k=-M}^{M} \left((1-2a)^2 + 4a^2\cos^2\left(\frac{\pi k}{M}\right) + (1-2a)4a\cos\left(\frac{\pi k}{M}\right)\right) \\
&= (1-2a)^2(2M+1) + 4a^2\left(1 + 2\sum_{k=1}^{M} 0.5\left(\cos\left(\frac{2\pi k}{M}\right) + 1\right)\right) \\
&+ (1-2a)4a\left(-1 + 2\sum_{k=0}^{M}\cos\left(\frac{\pi k}{M}\right)\right) \\
&= (1-2a)^2(2M+1) + 4a^2(1+M) + (1-2a)4a(-1) \\
&= (2 + 12a^2 - 8a)M + (1 + 16a^2 - 8a)
\end{aligned}$$

Thus the variance relation is

$$\begin{aligned}
\text{Var}\left[\frac{\hat{f}(\omega)}{f(\omega)}\right] &= \sum_{k=-M}^{M} \lambda_k^2/N \\
&= \underline{\underline{\big((2 + 12a^2 - 8a)M + (1 + 16a^2 - 8a)\big)}}
\end{aligned}$$

For the Tukey-Hanning window with a = 0.25 the variance relation becomes

$$\text{Var}\left[\frac{\hat{f}(\omega)}{f(\omega)}\right] = \frac{3M}{\underline{\underline{4N}}} \,,$$

which is in accordance with Table 7.2 on page 199.
*Question 4.*

For a Parzen window the variance relation is

$$\text{Var}\left[\frac{\hat{f}(\omega)}{f(\omega)}\right] = \frac{0.5391M}{N} \,.$$

I.e. the variance relation is unchanged for

$$\frac{0.5391M_p}{N} = \frac{3M_T}{4N} \Rightarrow$$
$$M_P = 1.39M_T = 1.39 \cdot 40 = \underline{\underline{56}}$$

# Solution 7.3

The spectral window is found as the Fourier transforme of the lag-window (the window in the time domain), i.e.

$$
\begin{aligned}
k(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \lambda(\tau)e^{i\tau\omega}d\tau \\
&= \frac{1}{2\pi}\int_{-1}^{1}(1-|\tau|)e^{-i\tau\omega}d\tau \\
&= \frac{1}{\pi}\int_{0}^{1}(1-\tau)\cos(\tau\omega)d\tau \\
&= \frac{1}{\pi\omega}\int_{0}^{1}\cos\tau\omega d\tau\omega - \frac{1}{\pi\omega^2}\int_{0}^{1}\tau\omega\cos(\tau\omega)d\tau\omega \\
&= \frac{1}{\pi\omega}\left[\sin(\tau\omega)\right]_0^1 - \frac{1}{\pi\omega^2}\left[\tau\omega\sin(\tau\omega)+\cos(\tau\omega)\right]_0^1 \\
&= \frac{1}{\pi\omega}\left(\frac{1}{\omega}(1-\cos(\omega)\right) = \frac{1-\cos\omega}{\pi\omega*^2} \\
&= \underline{\underline{\frac{2\sin^2(\omega/2)}{\pi\omega^2}}}
\end{aligned}
$$

## Solution 7.4

A 100(1-$\alpha$)% konfidence interval for $f(\omega)$ is given by

$$\left[\frac{\nu\hat{f}(\omega)}{\chi^2(\nu)_{1-\alpha/2}}, \frac{\nu\hat{f}(\omega)}{\chi^2(\nu)_{\alpha/2}}\right]$$

Taking the logarithm we get

$$\left[\frac{\nu(\omega)}{\chi^2(\nu)_{1-\alpha/2}} + \log(\hat{f}(\omega)), \frac{\nu(\omega)}{\chi^2(\nu)_{\alpha/2}} + \log(\hat{f}(\omega))\right]$$

which means that for a logarithmic scale the width of the confidence interval is independent of $\omega$.
For a Parzen window with N=400 and M=48 the degrees of freedom becomes

$$\nu \approx 3.71\frac{400}{48} \approx 31$$

Since the given figure for the spectrum is shown in decades it is suitable to use the confidence interval in logarithm to the base 10.

$$\log_{10}\left(\frac{31}{\chi^2(31)_{0.9}}\right) = \frac{31}{41.4} \approx -0.13$$
$$\log_{10}\left(\frac{31}{\chi^2(31)_{0.1}}\right) = \frac{31}{21.4} \approx 0.16$$

On the figure the ordinate values are

$$1 \rightarrow \log_{10}(1) = 0$$
$$2 \rightarrow \log_{10}(2) = 0.3010$$

This distance is about the value we estimated for the confidence interval (0.16+0.13=0.29), and we can therefore draw in an approximate confidence interval for the smoothed spectrum. We find that the theoretical spectrum is not everywhere inside a 80% confidence interval.

# Solution 7.5

*Question 1.*

Cross-covariance function

$$\gamma_{12}(\tau) = \text{Cov}[X_{1t}, X_{2t+\tau}] = \text{Cov}[\epsilon_{1t}, \beta_1\epsilon_{1t+\tau} + \beta_2\epsilon_{1t+\tau-1} + \epsilon_{2t}]$$
$$= \begin{cases} \beta_1\sigma_1^2 & \tau = 0 \\ \beta_2\sigma_1^2 & \tau = 1 \\ 0 & \text{otherwise} \end{cases}$$

*Question 2.*

Cross-spectrum:

$$f_{12}(\omega) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty}\gamma_{12}(\tau)e^{-i\omega\tau}$$
$$= \frac{\sigma_1^2}{2\pi}\left(\beta_1 + \beta_2\cos\omega - i\beta_2\sin\omega\right)$$

Co-spectrum:

$$c_{12}(\omega) = \frac{\sigma_1^2}{2\pi}\left(\beta_1 + \beta_2\cos\omega\right)$$

Quadrature spectrum:

$$q_{12}(\omega) = \frac{\sigma_1^2}{2\pi}\left(\beta_2\sin\omega\right)$$

Cross-amplitude spectrum:

$$\alpha_{12}(\omega) = \frac{\sigma_1^2}{2\pi}\left(\beta_1^2 + \beta_2^2\cos^2\omega + 2\beta_1\beta_2\cos\omega + \beta_2^2\sin^2\omega\right)$$
$$= \frac{\sigma_1^2}{2\pi}\left(\beta_1^2 + \beta_2^2 + 2\beta_1\beta_2\cos\omega\right)^{\frac{1}{2}}$$

Phase spectrum:

$$\phi_{12}(\omega) = \arctan\left(\frac{-\beta_2\sin\omega}{\beta_1 + \beta_2\cos\omega}\right)$$

*Question 3.*

For $\beta_2/\beta_1 = 1 \Leftrightarrow \beta_2 = \beta_1$ holds

$$\begin{aligned}
\alpha_{12}(\omega) &= \frac{\sigma_1^2}{2\pi}\beta_1 \left(2 + 2\cos\omega\right)^{\frac{1}{2}} = \frac{\sigma_1^2}{2\pi}\beta_1 \left(4\cos^2\frac{\omega}{2}\right)^{\frac{1}{2}} = \frac{\sigma_1^2}{\pi}\beta_1 \cos\frac{\omega}{2} \\
\phi_{12}(\omega) &= \arctan\left(\frac{-\sin\omega}{1+\cos\omega}\right) = \arctan\left(\frac{-2\sin\frac{\omega}{2}\cos\frac{\omega}{2}}{2\cos^2\frac{\omega}{2}}\right) \\
&= \arctan\left(\frac{\sin\frac{-\omega}{2}}{\cos\frac{-\omega}{2}}\right) = -\frac{\omega}{2}
\end{aligned}$$

For $\beta_2/\beta_1 = -1 \Leftrightarrow \beta_2 = -\beta_1$ holds

$$\begin{aligned}
\alpha_{12}(\omega) &= \frac{\sigma_1^2}{2\pi}\beta_1 \left(2 - 2\cos\omega\right)^{\frac{1}{2}} = \frac{\sigma_1^2}{2\pi}\beta_1 \left(4\sin^2\frac{\omega}{2}\right)^{\frac{1}{2}} = \frac{\sigma_1^2}{\pi}\beta_1 \sin\frac{\omega}{2} \\
\phi_{12}(\omega) &= \arctan\left(\frac{\sin\omega}{1-\cos\omega}\right) = \arctan\left(\frac{2\sin\frac{\omega}{2}\cos\frac{\omega}{2}}{2\sin^2\frac{\omega}{2}}\right) \\
&= \arctan\left(\frac{\cos\frac{\omega}{2}}{\sin\frac{\omega}{2}}\right) = \arctan(\cot\frac{\omega}{2}) \\
&= \arctan\left(\tan\left(\frac{\pi}{2}-\frac{\omega}{2}\right)\right) = \frac{\pi}{2}-\frac{\omega}{2}
\end{aligned}$$

The cross-amplitude and the phase spectrum is shown in Figure 1. The cross-amplitude spectrum shows that the covariance between the two processes are dominated by low frequencies if $\beta_2/\beta_1 = 1$ and by high frequencies if $\beta_2/\beta_1 = -1$. In generel if the cross-correlation is of same sign the covariances will be dominated by low frequencies.

The Phase spectrum shows that for $\beta_2/\beta_1 = 1$ the variation in $X_{2t}$ follows the variation in $X_{1t}$, while the opposite case occurs when $\beta_2/\beta_1 = -1$.

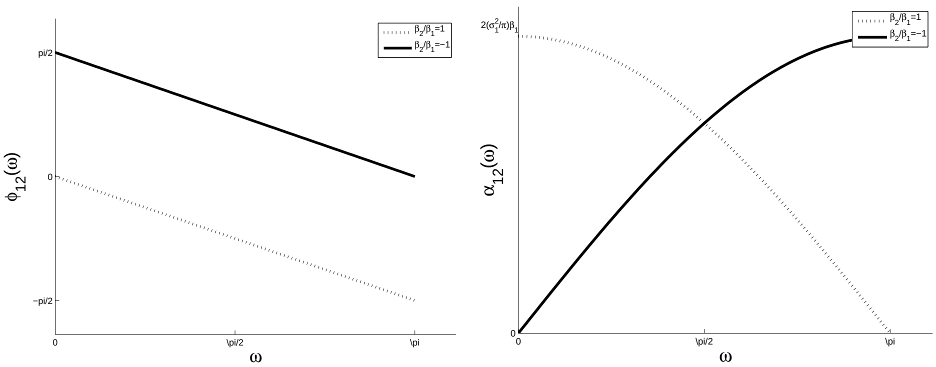


Figure 1: Left: Phase spectrum. Right: Cross-amplitude spectrum.