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Time Series Analysis
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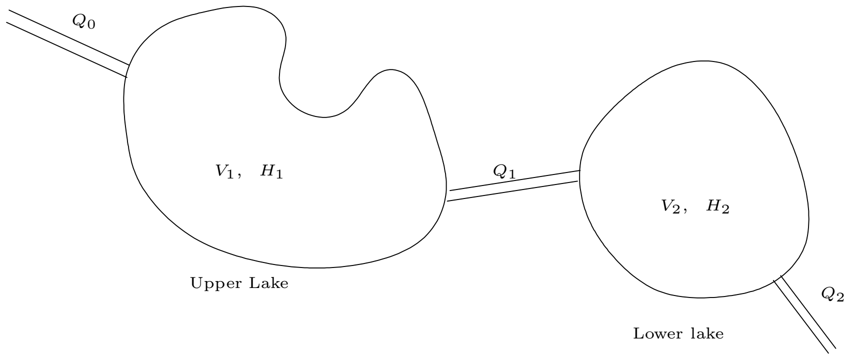
The dynamics of a Lake/River system[1]. Assignment related to Chapter 4 of the book

This assignment is inspired by the lake system along Mølleåen nearby DTU. The main purpose is the study the dynamics of the water level originating from the fact that two (major) lakes are found in the system. The constants given here are purely artificial and do not reflect the reality in any named lake.

In the following $V_.$ is the water volume, and $H_.$ the water level of the considered lake. $Q_.$ is the flow of the output from the lake.



1 Stationary points

In the following the time arguments are omitted. The water volumes in the two lakes are

$$V_1 = \int_0^{H_1} A_1(h)dh \qquad V_2 = \int_0^{H_2} A_2(h)dh$$

where H_1 and H_2 are the levels in the two lakes. The inflow to the first lake is denoted as Q_0 whereas

$$Q_1 = \sigma_1\sqrt{2gH_1} \qquad Q_2 = \sigma_2\sqrt{2gH_2}$$

[1]The assignment is found at `www.imm.dtu.dk/~hm/time.series.analysis`. This assignment is often skipped by non-engineering classes at the university.

are the flows between the lakes and outflow, respectively. Here σ_1 and σ_2 are constants characterizing the rivers (weirs and other objects making a flow resistance). The dynamics is embedded in the conservation of masses (volumes), i.e.

$$\frac{d}{dt}V_1 = Q_0 - Q_1 \qquad \frac{d}{dt}V_2 = Q_1 - Q_2$$

If a constant inflow, $\bar{Q}_0$, is applied then the situation will reach a steady state situation (in principle after infinite long time). In that situation the flows are all equal (due to no change in volumes).

Question 1

Show that the steady state levels are

$$\bar{H}_1 = \frac{1}{2g}\left(\frac{\bar{Q}_0}{\sigma_1}\right)^2 \qquad \bar{H}_2 = \frac{1}{2g}\left(\frac{\bar{Q}_0}{\sigma_2}\right)^2$$

Assume that the area (A) does not depend on the level (h) for both lakes. Now, introduce the quantities

$$\alpha = \sigma_1\sqrt{\frac{2g}{\bar{H}_1}} \qquad \beta = \sigma_2\sqrt{\frac{2g}{\bar{H}_2}}$$

then a linearized model (in the deviation away from the stationary values, and hence the new lower case variables) can be obtained:

$$A_1\dot{h}_1 = q_i - \alpha h_1 \qquad A_2\dot{h}_2 = \alpha h_1 - \beta h_2$$

or

$$\tau_1\dot{h}_1 + h_1 = q_i \qquad \tau_2\dot{h}_2 + h_2 = K h_1$$

where

$$\tau_1 = \frac{A_1}{\alpha} \qquad \tau_2 = \frac{A_2}{\beta} \qquad K = \frac{\alpha}{\beta}$$

Using the principle behind the Laplace transformation it is easily seen that the total model can summarized as:

$$\left(\tau_2\frac{d}{dt} + 1\right)\left(\tau_1\frac{d}{dt} + 1\right)h_2 = K q_i$$

or if the output $y(t) = h_2$ and the input $x(t) = q_i$ is applied as

$$\Big(\tau_2 \frac{d}{dt} + 1\Big)\Big(\tau_1 \frac{d}{dt} + 1\Big) y(t) = K x(t)$$

This model can also be expressed as

$$\tau_1 \tau_2 \frac{d^2}{dt^2} y(t) + (\tau_1 + \tau_2) \frac{d}{dt} y(t) + y(t) = K x(t)$$

Question 2

Verify the equations.

Assumed in the following that the numerical values

$$K = 2 \qquad \tau_1 = 1 \qquad \tau_2 = 1.5$$

can be applied.

2 Continuous time descriptions

Now consider the dynamic system described in continuous time with the differential equation:

$$\Big(\tau_2 \frac{d}{dt} + 1\Big)\Big(\tau_1 \frac{d}{dt} + 1\Big) y(t) = K x(t)$$

This is a model formulated in the time domain and with the differential operator

$$\frac{d}{dt}$$

Let

$$Y(s) = \mathcal{L}\{y(t)\}$$

denote the Laplace transform of $y(t)$.

Question 3

Show that the transfer function is given by:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{K}{(s\tau_1 + 1)(s\tau_2 + 1)} = \frac{\frac{K}{\tau_1 \tau_2}}{(s + \frac{1}{\tau_1})(s + \frac{1}{\tau_2})} = \frac{K}{\tau_1 \tau_2 \; s^2 + (\tau_1 + \tau_2) \; s + 1}$$

The Laplace transform is, as also indicated above, a very important tool for handling linear differential equations, cf. also Theorem 4.17 in [1]. Now we will use a numerical approach.

Assume that the following lines has been entered into Matlab (The time is assumed to be measured in days).

```
T1=1;
T2=1.5;
K=2;

Hs=tf(K,conv([T1 1],[T2 1]));
```

The characteristics of a dynamic system can determined by the roots of the denumerator (denoted as poles) and numerator (denoted as zeros).

Question 4

Find the poles and zeros of the system. Is the system stable?

The (numerical values for the) poles and zeros of a system can be determined in Matlab by the these commands

```
a=Hs.den{1};              % get the characteristic polynomial
roots(a)                  % determine the poles

b=Hs.num{1};              % get the numerator polynomial
roots(b)                  % determine the zeros
```

Another property of the Laplace transform is (see (4.97) in [1])

$$Y(s) = H(s)X(s)$$

which means that a response $y(t)$ from a input $x(t)$ can be determined by (multiply $H(s)$ and the Laplace transformed input $X(s)$ and determine) the inverse Laplace transform. Here we will pursuit a more numerical approach and determine the impulse response and step response by means of Matlab commands. The lines:

```
% Producing an impulse response
impulse(Hs); grid

% Producing a step response
step(Hs); grid;
```

produces the impulse response and step response function, respectively.

Question 5

Use Matlab to plot the impulse and step responses. What is the steady state gain of the system?

It is clear that variation of the water level is highly influenced by the fact that the two lakes are connected in series. For comparison we shall now try to approximate the system by a first order model.

Hence, consider the approximation of the lake system by the transfer function:

$$H_1(s) = \frac{K}{s\tau_2 + 1} \qquad (1)$$

Question 6

Use Matlab to plot the impulse and step responses simultaneously for the first order approximation and the actual system. Comment on the findings.

Periodic signal (and others) can be described by its contents of components (of e.g. harmonic functions) at different frequencies.

A dynamic system is often charecterized by the way it transforms the different frequencies.

Consider for example

$$x(t) = sin(\omega t) \quad \omega = \frac{2\pi}{T} \quad T = 6.3$$

then the response can be found (numerically) and plot by the following Matlab commands:

```
t=0:80;
x=sin(w*t);
y=lsim(Hs,x,t);
plot(t,x,t,y); grid;
```

Question 7

Use Matlab as described above to plot the harmonic input and the resulting output. What happens with the amplitude? What is the phase shift from input to output? Only approximate values in should be provided from reading the plot.

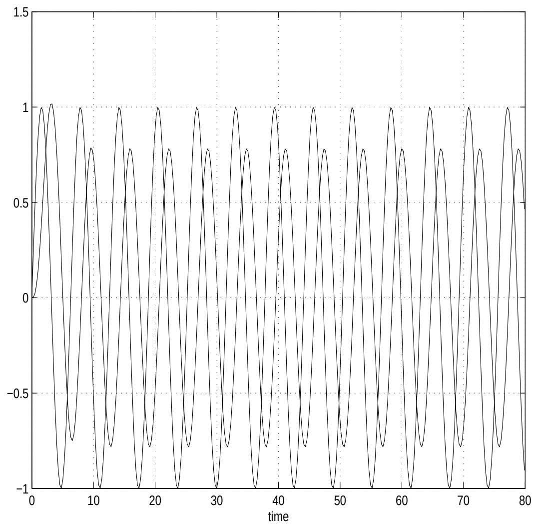


Figure 1: Plot of $x(t)$ (dashed) and y(t) (solid) for a harmonic input. Note the transient response in the start of the series.

From (4.107) in [1] the frequency response can be determined from $H(s)$ simply by substituting s with $i\omega$

$$\mathcal{H}(\omega) = H(i\omega)$$

eq1 which (for each ω) is a complex number. This is typically plotted as the length of H (amplitude) and as the angle of H (phase, phase shift) as a function of ω. Together these two plos is called a *Bode plot* for the system.

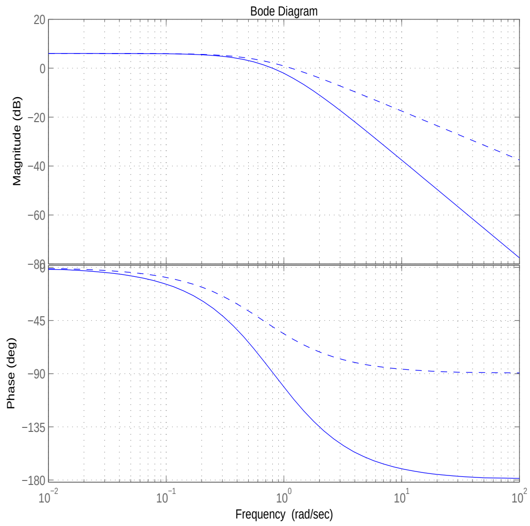


Figure 2: Bode plot (unit: rad/day)

Question 8

For the actual case, where $\omega = 0.9973$, show that the frequency response function is

$$\mathcal{H}(\omega) = 0.7870\ exp(-i\ 101.2(deg))$$

The Matlab lines producing the bode plot (for the lake system only) is simply:

```
bode(Hs);
grid;

T=6.3; w=2*pi/T;

[mag,ph]=bode(Hs,w)
```

Question 9

Plot the amplide and phase function, and comment on the results.

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3 Discrete time models

The lake system can also be described in discrete time. The transformation of a description in continuous time into a similar in discrete time is a complex mapping and beyond the scope of this exercise. Here we will simply just apply Matlab and work on the results.

In discrete time the Z-transform has the same role as the Laplace transform in continuous time. Actually it can be regarded as a special case of the Laplace transform just applied for discrete time signals and systems. As shown in the lecture notes the relation between the complex variable, s in the Laplace transform and z in the Z-transform is simply:

$$z = e^{sT}$$

where T is the sampling period (see also (4.112) in [1]).

Let $Y(z)$ denote the Z transform of a discrete time signal y_t. One of the most prominent properties of the Z-transform is that

$$\mathcal{Z}\{y_{t+1}\} = zY(z) \qquad \mathcal{Z}\{y_{t-1}\} = z^{-1}Y(z)$$

The choice of sampling period has to be done in accordance with the application (signal processing, time series analysis, system identification or control). Since the fasted time constant (in this case) is 1 *day* we will choose $T = 0.3$.

The discrete time description of the lake system can in terms of the transfer function (in the Z domain) be found by the matlab commands

```
Ts=0.3;
Hd=c2d(Hs,Ts,'foh') % transform the description into discrete time
transfer function:
0.01768 z^2 + 0.06251 z + 0.01377
---------------------------------
      z^2 - 1.56 z + 0.6065

Sampling time: 0.3
```

In other words the transfer function is

$$H(z) = \frac{0.01768z^2 + 0.06251z + 0.01377}{z^2 - 1.56z + 0.6065}$$

Often we are using the back shift operator B and forward shift operator F where

$$By_t = y_{t-1} \qquad Fy_t = y_{t+1}$$

We can the describe the system in the time domain as

$$(F^2 - 1.56F + 0.6065)y_t = (0.01768F^2 + 0.06251F + 0.01377)x_t$$

or as

$$y_{t+2} - 1.56\ y_{t+1} + 0.6065\ y_t = 0.01768\ x_{t+2} + 0.06251\ x_{t+1} + 0.01377\ x_t$$

Instead of using forward notation we can equally well use the backward notation:

$$(1 - 1.56\ B + 0.6065\ B^2)\ y_t = (0.01768 + 0.06251\ B + 0.01377\ B^2)\ x_t$$

or as:

$$y_t - 1.56\ y_{t-1} + 0.6065\ y_{t-2} = 0.01768\ x_t + 0.06251\ x_{t-1} + 0.01377\ x_{t-2}$$

which again can be written as:

$$y_t = 1.56\ y_{t-1} - 0.6065\ y_{t-2} + 0.01768\ x_t + 0.06251\ x_{t-1} + 0.01377\ x_{t-2}$$

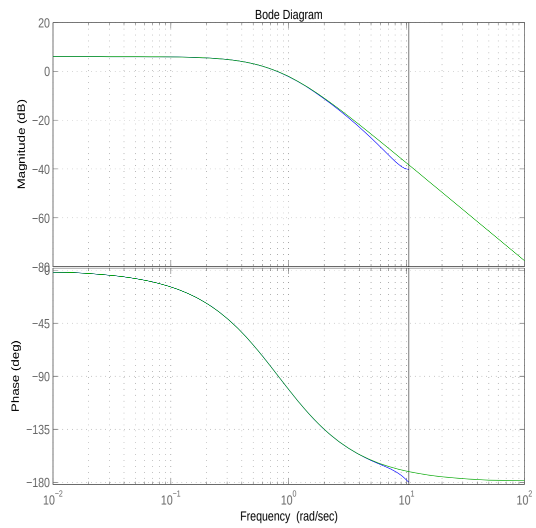

Figure 3: Bode plot for the time lake system (solid). Unit is rad/day

The step response and the frequency response (see Figure 3) (bode plot) for both the discrete and continuous time transfer function can easily be produced in Matlab with the lines

```
step(Hd,Hs);
grid

bode(Hd,Hs);
grid
```

Question 10

Plot the discrete time step and frequency functions. Compare with the continuous version of the same functions.

The frequency response, i.e. the gain and phase shift of a harmonic function for a certain frequency, can be determinated according to (4.64) in [1] as

$$\mathcal{H}(\omega) = H(e^{i\omega T_s})$$

Question 11

Show for the considered frequency[2], either by the bode plot or from H directly that

$$\mathcal{H}(\omega) = 0.7811 exp(-i\ 101.2(deg))$$

A simulation of the deterministic system can be obtained by the following Matlab commands:

```
nstp=300;
i=0:nstp;
xd=sin(w*Ts*i);
a=[1 -1.5595 0.6065];
b=[0.0177 0.0625 0.0138];
yd=dlsim(b,a,xd);
plot(i,xd,'+--',i,yd,'+-');
```

and the result can be seen in Figure 4.

From this it is possible to see the gain and the phase shift for the considered frequency.

[2]Note, in (4.64) ω is the normalized angular frequency.

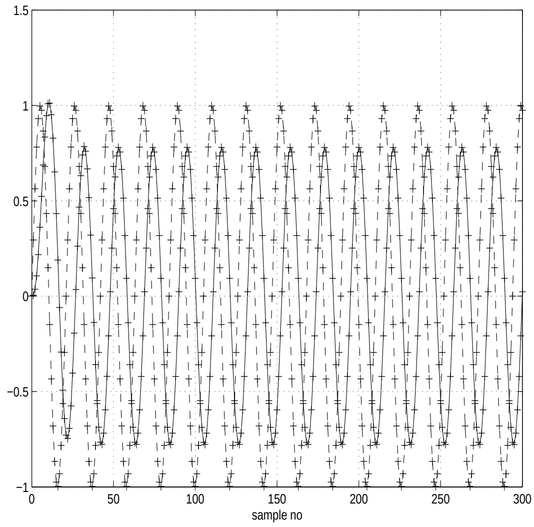

Figure 4: Plot of $x(i)$ (dashed) and y(i) (solid) for a harmonic input.

References

[1] H. Madsen. *Time Series Analysis*. Chapman Hall/CRC, 2007.