Time Series Analysis

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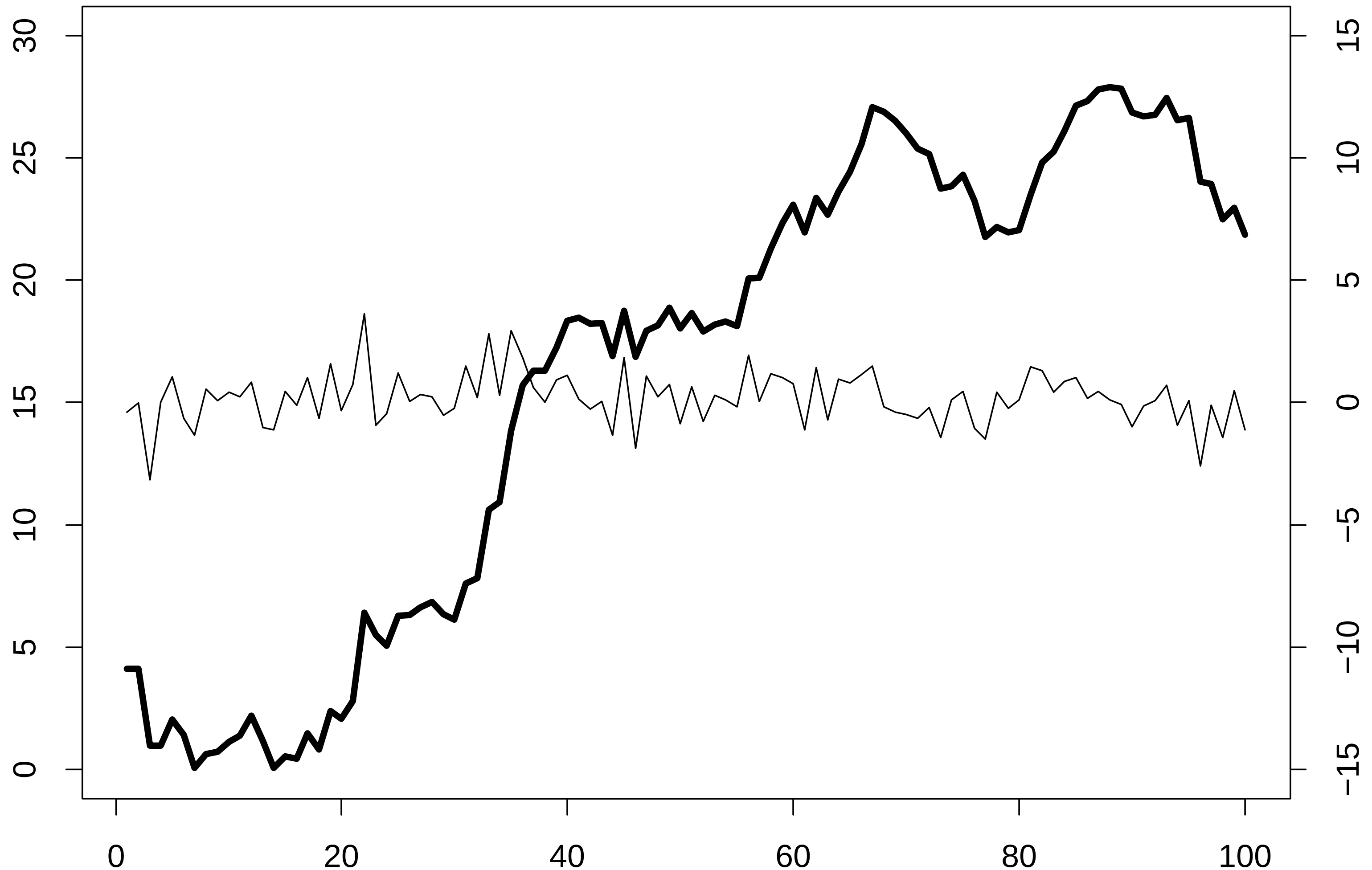
Outline of the lecture

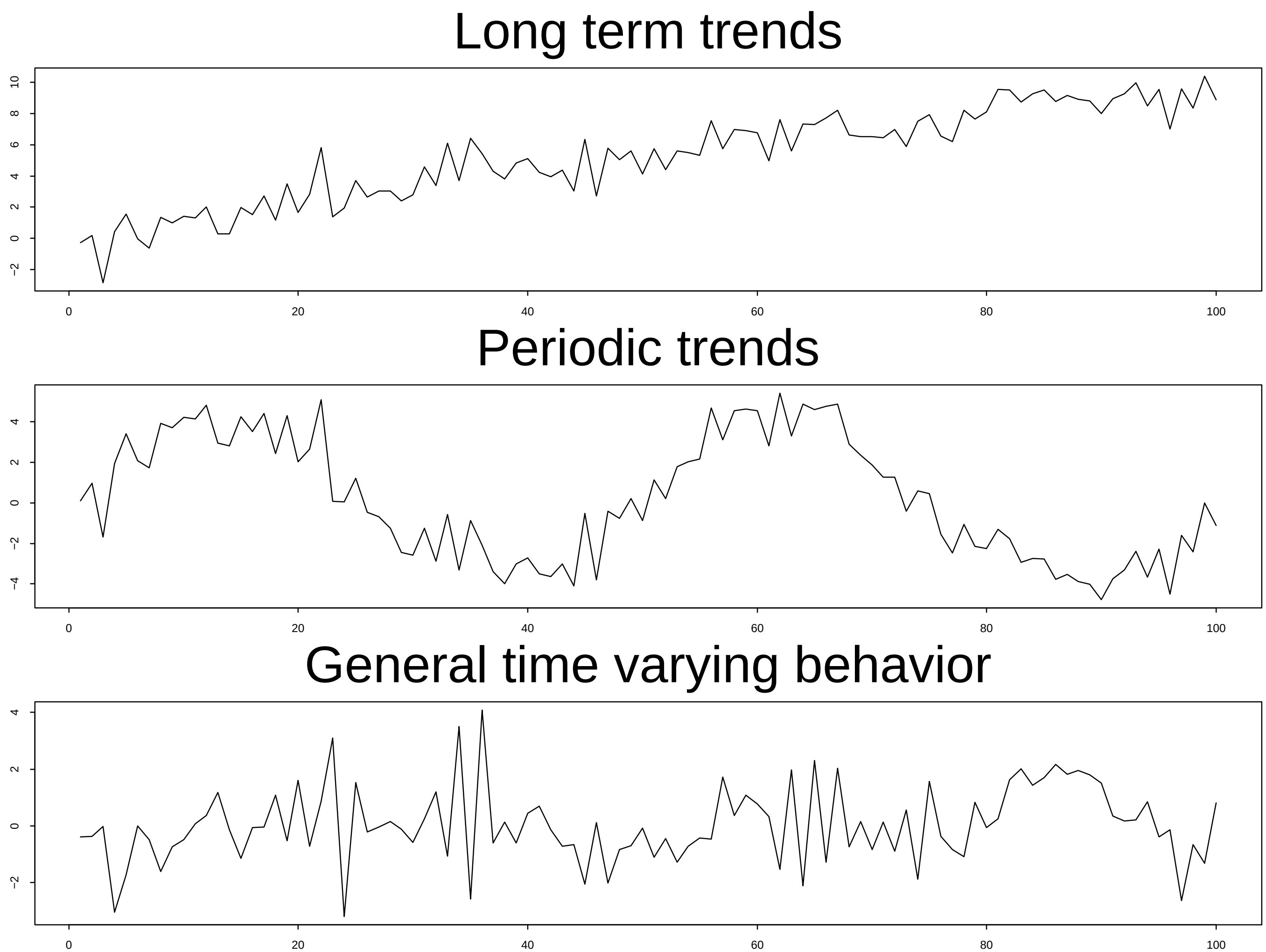
Stochastic processes, 2nd part:

- Non-stationary models, Sec. 5.6
- Optimal Prediction, Sec. 5.7

Non-stationary series



Some types of non-stationarity



The $ARIMA(p, d, q)$-process

- An $ARMA(p, q)$ model for:

$$W_t = \nabla^d Y_t = (1 - B)^d Y_t$$

where $\{Y_t\}$ is the series

- That is:

$$\phi(B)\nabla^d Y_t = \theta(B)\varepsilon_t$$

- If we consider stationarity:

$$\phi(z^{-1})(1 - z^{-1})^d = 0$$

i.e. d roots in $z = 1 + 0i$, and the rest inside the unit circle

The $(p, d, q) \times (P, D, Q)_s$ seasonal process

- A multiplicative (stationary) $ARMA(p, q)$ model for:

$$W_t = \nabla^d \nabla_s^D Y_t = (1 - B)^d (1 - B^s)^D Y_t$$

where $\{Y_t\}$ is the series

- That is:

$$\phi(B)\Phi(B^s)\nabla^d \nabla_s^D Y_t = \theta(B)\Theta(B^s)\varepsilon_t$$

- If we consider stationarity:

$$\phi(z^{-1})(1 - z^{-1})^d (1 - z^{-s})^D = 0$$

i.e. d roots in $z = 1 + 0i$, $D \times s$ roots on the unit circle, and the rest inside the unit circle

The case $d = D = 0$; stationary seasonal process

- General:

$$\phi(B)\Phi(B^s)Y_t = \theta(B)\Theta(B^s)\varepsilon_t$$

- Example:

$$(1 - \Phi B^{12})Y_t = \varepsilon_t$$

- Which can also be written:

$$Y_t = \Phi Y_{t-12} + \varepsilon_t$$

  i.e. Y_t depend on $Y_{t-12},\ Y_{t-24},\ \ldots$ (thereof the name)

- How would you think that the auto covariance function looks?
- Take a look at Example 5.10 also.

Prediction

- At time t we have observations $Y_t, Y_{t-1}, Y_{t-2}, Y_{t-3}, \ldots$
- We want a prediction of Y_{t+k}, where $k \geq 1$
- If we want to minimize the expected squared error the optimal prediction is the conditional expectation:

$$\widehat{Y}_{t+k|t} = E[Y_{t+k}|Y_t, Y_{t-1}, Y_{t-2}, \cdots]$$

Example – prediction in the $AR(1)$ model

- We write the model like $Y_{t+1} = \phi Y_t + \varepsilon_{t+1}$ (note the sign on ϕ)
- 1-step prediction:

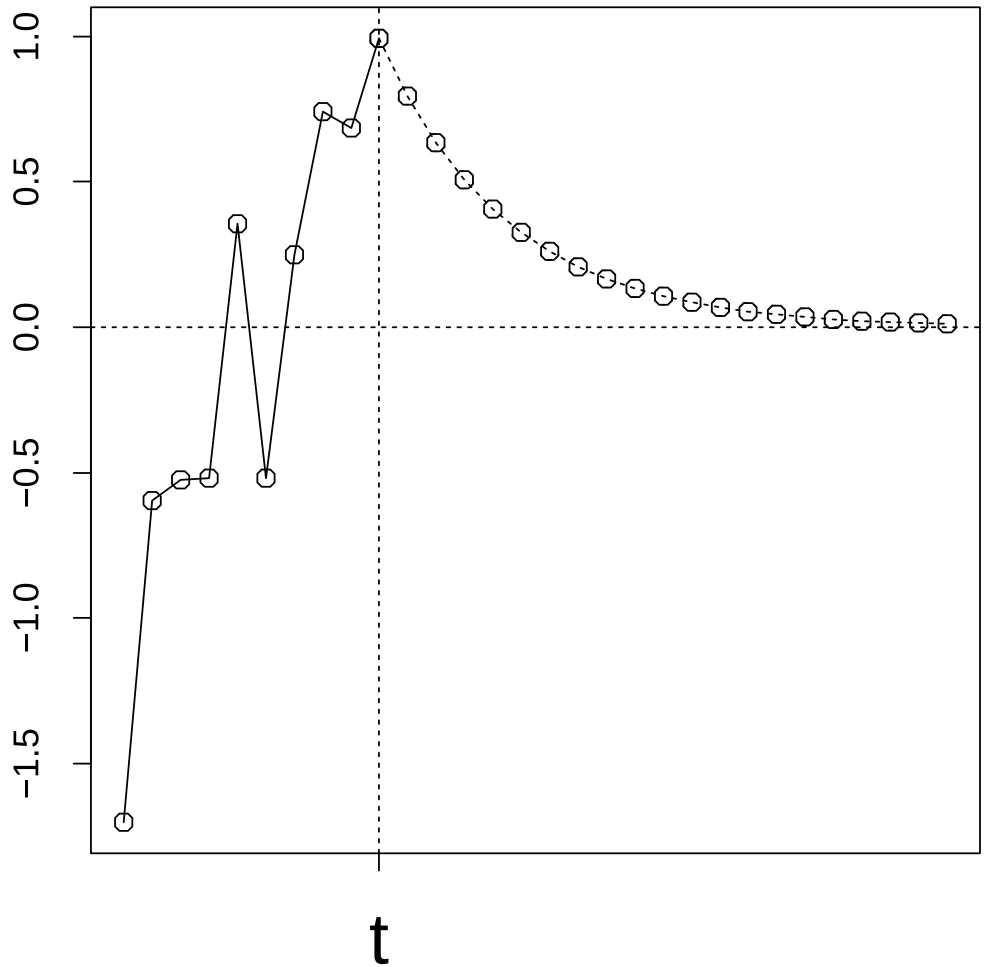
$$\begin{aligned}\widehat{Y}_{t+1|t} &= E[Y_{t+1}|Y_t, Y_{t-1}, \cdots] = E[\phi Y_t + \varepsilon_{t+1}|Y_t, Y_{t-1}, \cdots] \\ &= \phi Y_t + 0 = \phi Y_t\end{aligned}$$

- 2-step prediction:

$$\begin{aligned}\widehat{Y}_{t+2|t} &= E[Y_{t+2}|Y_t, Y_{t-1}, \cdots] = E[\phi Y_{t+1} + \varepsilon_{t+2}|Y_t, Y_{t-1}, \cdots] \\ &= \phi \widehat{Y}_{t+1|t} + 0 = \phi^2 Y_t\end{aligned}$$

- k-step prediction: $\boxed{\widehat{Y}_{t+k|t} = \phi^k Y_t}$

Example – prediction in $Y_t = 0.8Y_{t-1} + \varepsilon_t$



Variance of prediction error for the $AR(1)$-process

Prediction error:

$$e_{t+k|t} = Y_{t+k} - \widehat{Y}_{t+k|t} = Y_{t+k} - \phi^k Y_t$$

$$\begin{aligned}
Y_{t+k} &= \phi Y_{t+k-1} + \varepsilon_{t+k} \\
&= \phi(\phi Y_{t+k-2} + \varepsilon_{t+k-1})\varepsilon_{t+k} \\
&= \phi^2 Y_{t+k-2} + \phi\varepsilon_{t+k-1} + \varepsilon_{t+k} \\
&= \phi^2(\phi Y_{t+k-3} + \varepsilon_{t+k-2}) + \phi\varepsilon_{t+k-1} + \varepsilon_{t+k} \\
&= \phi^3 Y_{t+k-3} + \phi^2\varepsilon_{t+k-2} + \phi\varepsilon_{t+k-1} + \varepsilon_{t+k} \\
&\vdots \\
&= \phi^k Y_t + \phi^{k-1}\varepsilon_{t+1} + \phi^{k-2}\varepsilon_{t+2} + \ldots + \phi\varepsilon_{t+k-1} + \varepsilon_{t+k}
\end{aligned}$$

Variance of prediction error for the $AR(1)$-process

Variance of prediction error:

$$\begin{aligned} V[e_{t+k|t}] &= V[\phi^{k-1}\varepsilon_{t+1} + \phi^{k-2}\varepsilon_{t+2} + \ldots + \phi\varepsilon_{t+k-1} + \varepsilon_{t+k}] \\ &= (\phi^{2(k-1)} + \phi^{2(k-2)} + \ldots + \phi^2 + 1)\sigma_\varepsilon^2 \end{aligned}$$

$(1-\alpha) \times 100\%$ prediction interval:

$$\widehat{Y}_{t+k|t} \pm u_{\alpha/2}\sqrt{V[e_{t+k|t}]}$$

$u_{\alpha/2}$ is the $\alpha/2$-quantile in the standard normal distribution

k-step prediction in $ARMA(p,q)$–models

The process:

$$Y_{t+k} + \phi_1 Y_{t+k-1} + \cdots + \phi_{p+d} Y_{t+k-p} = \varepsilon_{t+k} + \theta_1 \varepsilon_{t+k-1} + \cdots + \theta_q \varepsilon_{t+k-q}$$

Using conditional expectation on both sides we get:

$$\begin{aligned} \widehat{Y}_{t+k|t} &= -\phi_1 \widehat{Y}_{t+k-1|t} - \cdots - \phi_p \widehat{Y}_{t+k-p|t} \\ &\quad + \widehat{\varepsilon}_{t+k|t} + \theta_1 \widehat{\varepsilon}_{t+k-1|t} + \cdots + \theta_q \widehat{\varepsilon}_{t+k-q|t} \end{aligned}$$

This results in a recursive method for calculating the predictions – how would you find $\varepsilon_{t+k-q|t}$?

Inverse form

For an invertible process, the π-weights

$$\varepsilon_t = Y_t + \pi_1 Y_{t-1} + \pi_2 Y_{t-2} + \cdots$$

goes to zero sufficiently fast and only recent values of the process is needed.

Variance of prediction error

Process written with ψ-weights:

$$Y_{t+k} = \varepsilon_{t+k} + \psi_1 \varepsilon_{t+k-1} + \cdots + \psi_k \varepsilon_t + \psi_{k+1} \varepsilon_{t-1} + \cdots$$

k-step prediction:

$$\widehat{Y}_{t+k|t} = \psi_k \varepsilon_t + \psi_{k+1} \varepsilon_{t-1} + \cdots$$

k-step prediction error:

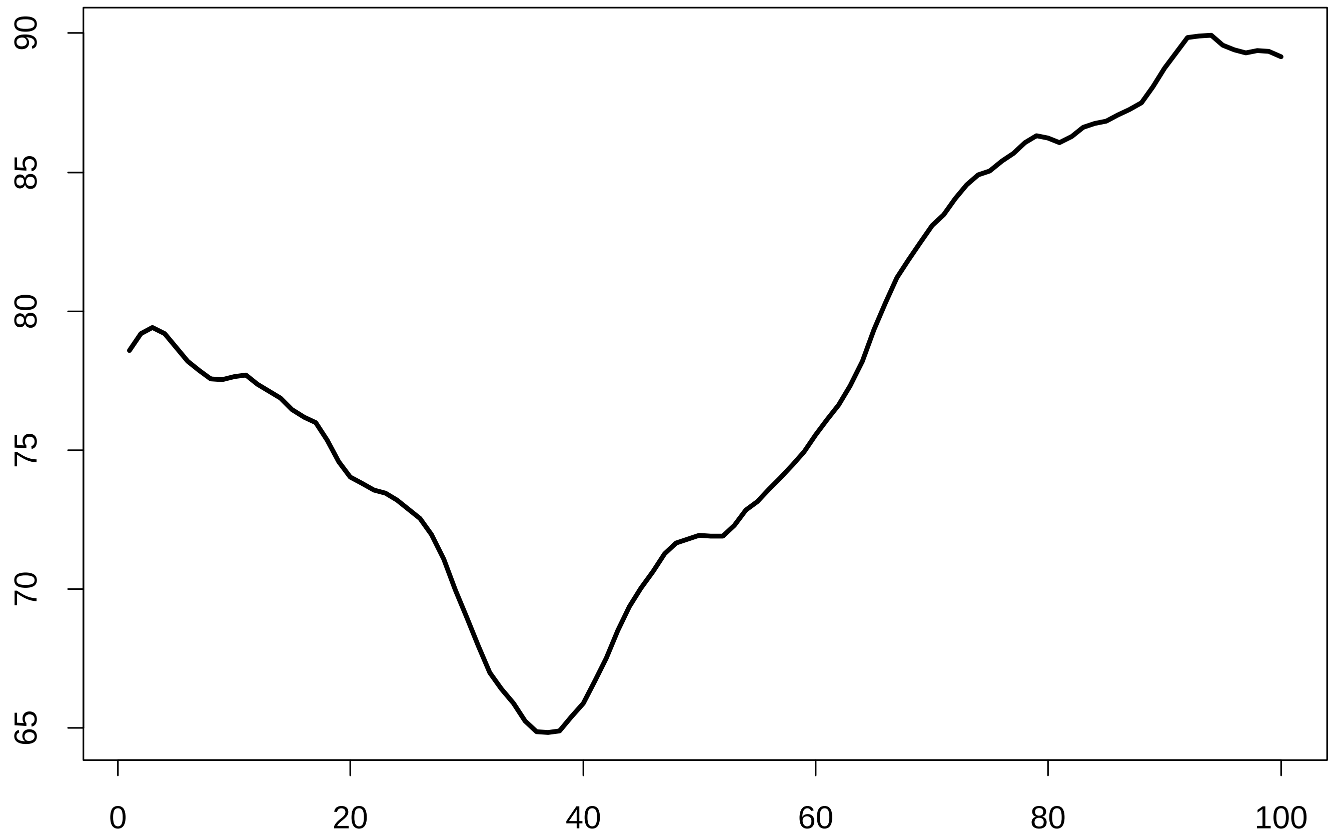
$$e_{t+k|t} = Y_{t+k} - \widehat{Y}_{t+k|t} = \varepsilon_{t+k} + \psi_1 \varepsilon_{t+k-1} + \cdots + \psi_{k-1} \varepsilon_{t+1}$$

Variance of k-step prediction error:

$$\sigma_k^2 = (1 + \psi_1^2 + \cdots + \psi_{k-1}^2) \sigma_\varepsilon^2$$

Prediction of bond prices

$$(1 - 1.274B + 0.3867B^2)\nabla Y_t = \varepsilon_t; \quad \sigma_\varepsilon^2 = 0.201^2; \quad \mu_Y = 84.3$$



Prediction of bond prices

- Price last 6 days: $\ldots, 90.79, 89.90, 88.88, 87.98, 87.41, 87.16$
- Prediction of price in two days:

$$\begin{aligned}
\varphi(B) &= \phi(B)\nabla = (1+\phi_1 B + \phi_2 B^2)(1-B) \\
&= 1 + (\phi_1 - 1)B + (\phi_2 - \phi_1)B^2 - \phi_2 B^3 \\
\widehat{Y}_{t+1|t} &= (1-\phi_1)Y_t + (\phi_1 - \phi_2)Y_{t-1} + \phi_2 Y_{t-2} = 87.07 \\
\widehat{Y}_{t+2|t} &= (1-\phi_1)\widehat{Y}_{t+1|t} + (\phi_1 - \phi_2)Y_t + \phi_2 Y_{t-1} = \underline{\underline{87.05}}
\end{aligned}$$

- Variance of prediction error:

$$V[\varepsilon_{t+2} + (1-\phi_1)\varepsilon_{t+1}] = (1 + (1-\phi_1)^2)\sigma_\varepsilon^2 = 0.500^2$$

- 95% prediction interval: $87.05 \pm 1.96 \cdot 0.50 = [86.07;\ 88.02]$