Markov Models in Discrete and Continuous Time for Hourly Observations of Cloud Cover

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(Manuscript received 8 May 1984, in final form 18 February 1985)

ABSTRACT

An analysis of 15 years of hourly observations of cloud cover at an airport location near Copenhagen, Denmark shows that the main part of the variations can be described by first-order homogeneous Markov models. Models in both discrete and continuous time are considered. Special emphasis is laid upon the representation of the variation by a Markov model in continuous time. The physical restrictions for transitions of cloud cover are investigated and it is shown that the natural restrictions imply a very simple structure for the matrix of transition rates corresponding to the embedded Markov process in continuous time. The maximum likelihood estimate of the matrix of transition rates is found by numerical methods and an indication of the estimation error under the model is given.

1. Introduction

The purpose of the present paper is to discuss the development of a simple Markov model for the variations in cloud cover. A secondary objective of the discussion will concentrate on the advantage of dealing with Markov models in continuous time.

It is well-known that many meteorological observations show correlations in time—the so-called autocorrelation. The autocorrelation is due to the inertia of variations according to the dynamics of the physical system. For a given system the autocorrelation between two successive observations will of course depend on their time distance. Gringorten (1966) found that the hour-to-hour correlation is around 0.95 for the main part of the surface weather observations taken at Minneapolis, Minnesota. The hour-to-hour correlation of weather observations is further discussed by Lund and Grantham (1979), who developed a model for recurrence probabilities of several weather events. In the present paper, the autocorrelations of cloud cover are investigated, and it is shown that the correlations and the variations can be described rather well by first-order homogeneous Markov models. An important characteristic of this Markov model is that the conditional probability of a change in cloud cover is dependent on the actual cloud cover only.

Markov models may be used to describe a great variety of climatological phenomena. A range of Markov models has been used in the past to describe rainfall occurrence; for example, Coe and Stern (1982) have used first- and second-order Markov chain models in which the parameters vary with the time of the year. Roldan-Cañas *et al.* (1982) have used first-order Markov models to describe the variation of the horizontal direction for daily wind. Generally, only Markov models in discrete time have been considered.

This paper considers Markov models for cloud cover in discrete as well as in continuous time. Both models are based on discrete time sampled data. The model in continuous time is shown to be attractive for several reasons, particularly because the matrix of transition rates belonging to the embedded Markov process in continuous time has a very simple structure which is due to the physical conditions for transitions of cloud cover. Furthermore it is our belief that the continuous time formulation of Markov models can be attractive in many other situations. Thus, beyond the discussion of the elaboration of a Markov model for cloud cover, the purpose of the present paper is to describe the formulation of Markov models in continuous time and the estimation of the matrix of transition rates, given a specific structure of the matrix.

Important factors in the energy balance at the earth's surface are highly affected by cloud cover variations. Global radiation is an example. The global radiation at the surface of the earth is a reduced part of the radiation from the sun incident to the top of the atmosphere due to reflection back to space and absorption by clouds, water vapor, ozone, aerosols and air molecules. Among these effects, the effect due to clouds is the primary one. Following Paltridge and Platt (1976) variations in global radiation due to variations in water vapor and aerosols are likely to be of the magnitude 3–5%. Therefore, a model for the variations of cloud cover is of interest in connection with parametrization of the variation of, e.g., global radiation. The effect of clouds on the radiation is very complicated, but several authors have established empirical models of the static relationship; Reed (1982) and Atwater and Ball (1978) are some examples.

2. The data

The investigation is based on 15 years of hourly cloud cover observations, taken at the synoptic observation station at Værløse Airport about 15 km northwest of Copenhagen, Denmark. The observations are total cloud cover in Oktas in accordance with WMO code (WMO, 1972, 1974). Thus, the cloud cover is measured as an integer between 0 and 9, where 0 corresponds to completely clear sky, 8 to completely overcast sky, and the observation 9 indicates that the cloud cover is unobservable which usually is the case in foggy weather or heavy snowfall. The observations are based on a subjective evaluation, and as such they may be encumbered with judgemental errors.

3. Empirical description of the seasonal variation

The seasonal variation of the cloud cover mean is illustrated in Fig. 1. The mean is calculated as the average over the 24 hours of the day and over the 15 years.

Although each data point thus represents 360 individual observations the graph is not very smooth. The well-known correlation between hourly observations implies a correlation between the 24 observations on each day, such that the effective number of observations behind the calculation is far less than 360.

Despite this random variation there is, however, a strong indication of a seasonal variation of the average cloud cover. It is seen from the figure that the winter is far more cloudy than the summer. Further the figure indicates that it is more cloudy in July than in the surrounding months.

In order to illustrate the annual variation of the *distribution* of cloud cover, Fig. 2 shows the distribution in January and June. These two distributions are extremes compared to the distributions of the remainder of the months where the distributions vary smoothly between the January and June distributions.

The diurnal variation of cloud cover is not shown, but we have found that the diurnal variation consists of a higher frequency of clear sky categories (0–1) at night than during the day. Conversely the frequency of middle categories is smaller at night than during the day. The frequency of overcast sky (7–8) is nearly constant during day and night. The result of the diurnal variation in the distribution of cloud cover is that on the average the day is more cloudy than the night. However, the diurnal variation is much smaller than the seasonal variation. Therefore as a first approximation the diurnal variation will not be considered in the analysis.

4. Hourly correlations

The persistence of a climatic event can be illustrated by the correlations between hourly observations with

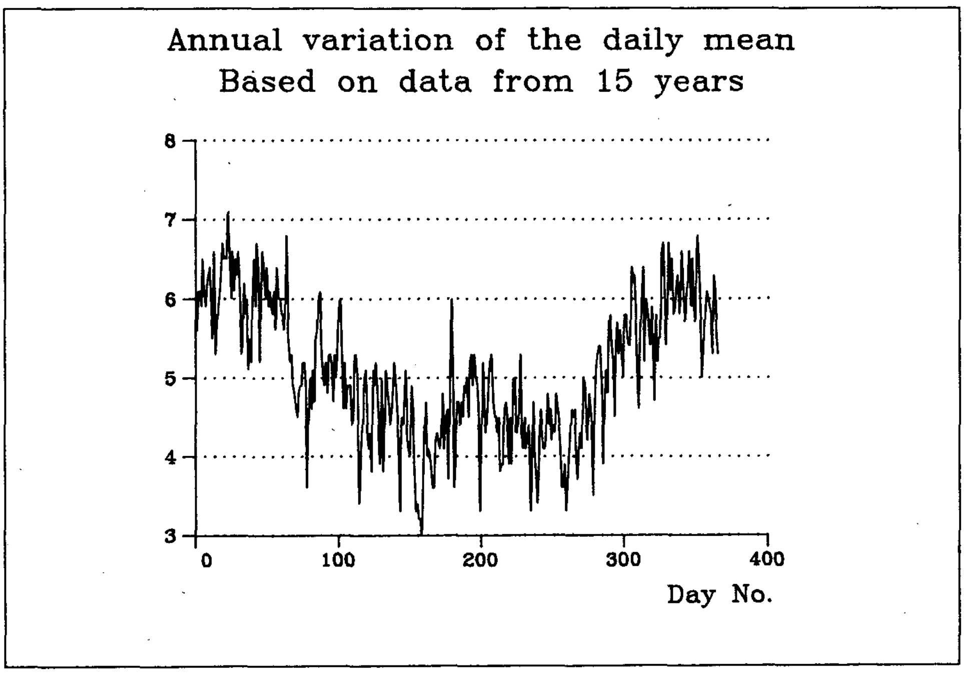

FIG. 1. Annual variation of the mean.

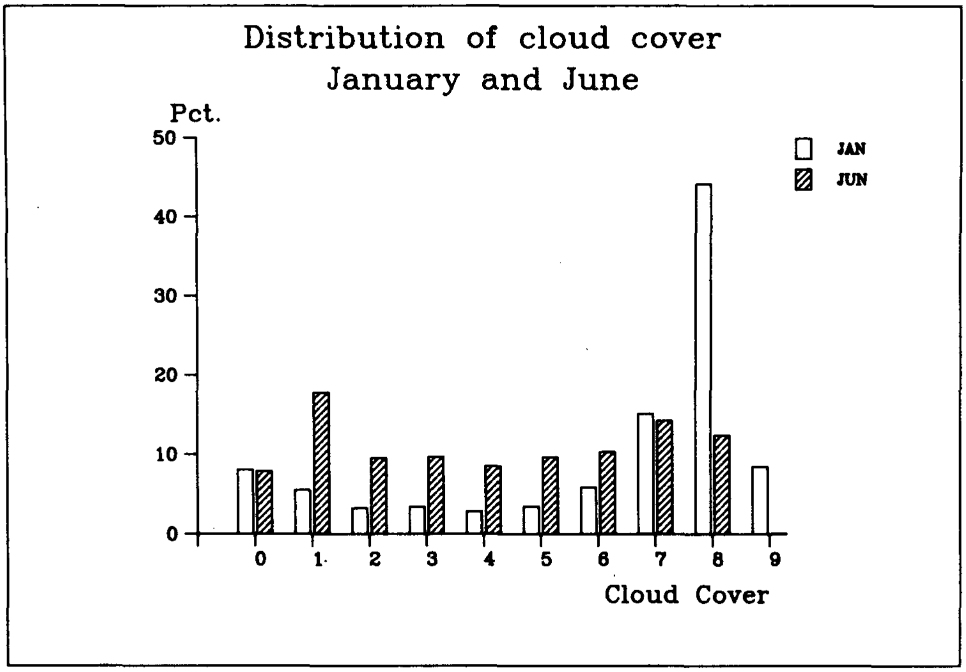

FIG. 2. Distribution of cloud cover. January and June.

various time-lags. The correlations corresponding to all time-lags are given by the *autocorrelation function* (see, e.g., Box and Jenkins, 1976). The estimated autocorrelation function, based on all observations, is shown in Fig. 3.

It is seen that due to the large number of observations the estimated autocorrelation function is rather smooth. The correlation corresponding to lag 1 hour (hour-to-hour correlation) is $\rho = 0.92$, which is in agreement with the generally established high hourly

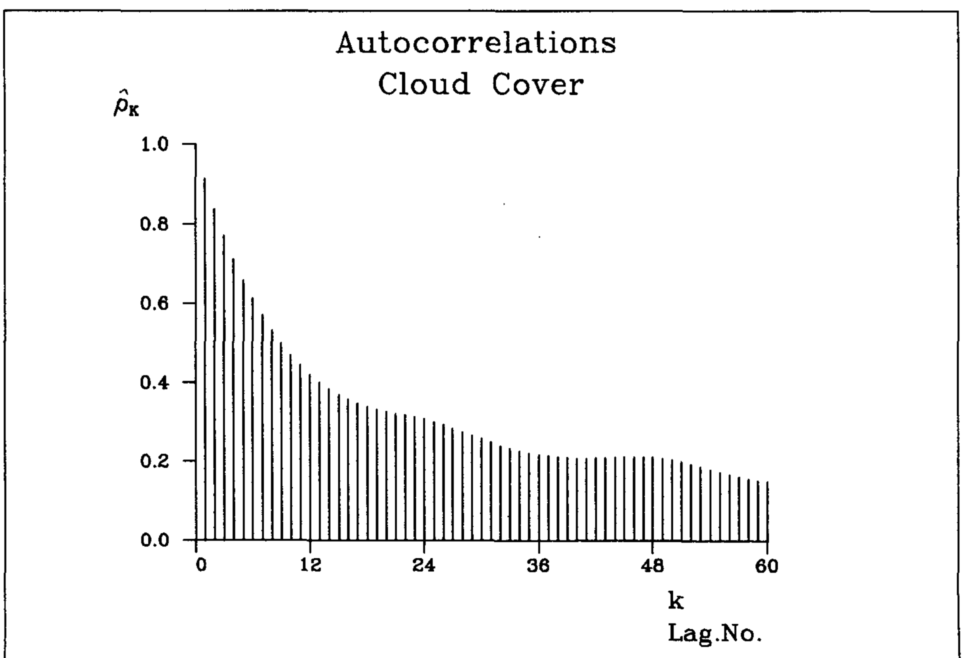

FIG. 3. Estimated autocorrelation function.

correlation between weather events. Further, it is seen that the autocorrelation dies out almost exponentially, i.e., the correlation corresponding to a lag of k hours is approximately equal to ρ^k, with ρ as above. But the exponential decay of the autocorrelation function is disturbed by a weak 24-hour oscillation, which most likely is due to the diurnal variation previously mentioned.

The dominating exponential decay of the autocorrelation function indicates that most of the statistical information about the development of the cloud cover in the next hours is contained simply in the present observation of cloud cover.

A further and distinct indication of the fact that nearly all the information about the future is contained in only the present observation, is given by the *partial autocorrelation function.* The value of the partial autocorrelation at lag k represents the additional information contained in the observation of the cloud cover k hours before the present observation, given the information from the observations in the intermediate hours. The estimated partial autocorrelation function, based on all observations, is shown in Fig. 4. Since the partial autocorrelation for lags greater than one hour is very small compared with the partial autocorrelation at lag one, the conclusion is reestablished that most of the information about the development of cloud cover is contained in the present observation.

Stochastic processes characterized by the condition that all the information about the future is contained in the present observation, are called first-order Markov processes. Such processes with parameters that do not vary with time are called homogeneous; otherwise they are called inhomogeneous.

In order to describe the seasonal variation we have chosen to set up a homogeneous Markov model for every month in the year. As a consequence of the restriction of the investigation to homogeneous Markov models only, the diurnal variation and the weak 24-hour oscillation in the autocorrelation function are neglected. This approach serves the main purpose of the present paper, which is to describe the estimation of parameters of the embedded Markov process in continuous time given a specific structure of the process. This estimation is more simple in the homogeneous case, and, further, the main part of the variation and correlation is fairly well described by the homogeneous model.

An improved first order Markov model must include time-varying parameters. In Section 8 we shall briefly comment upon the possibility of using such inhomogeneous Markov models.

5. Discrete time Markov model

The cloud cover is measured by the integers 0 through 9. Because of this finite number of states, it is reasonable to try to describe the variations by a Markov chain model.

The observations will be denoted

$$X_1, X_2, \ldots, X_N$$

where $X_t \in \{0, 1, \ldots, 9\}$ denotes the state of the cloud cover category at time t.

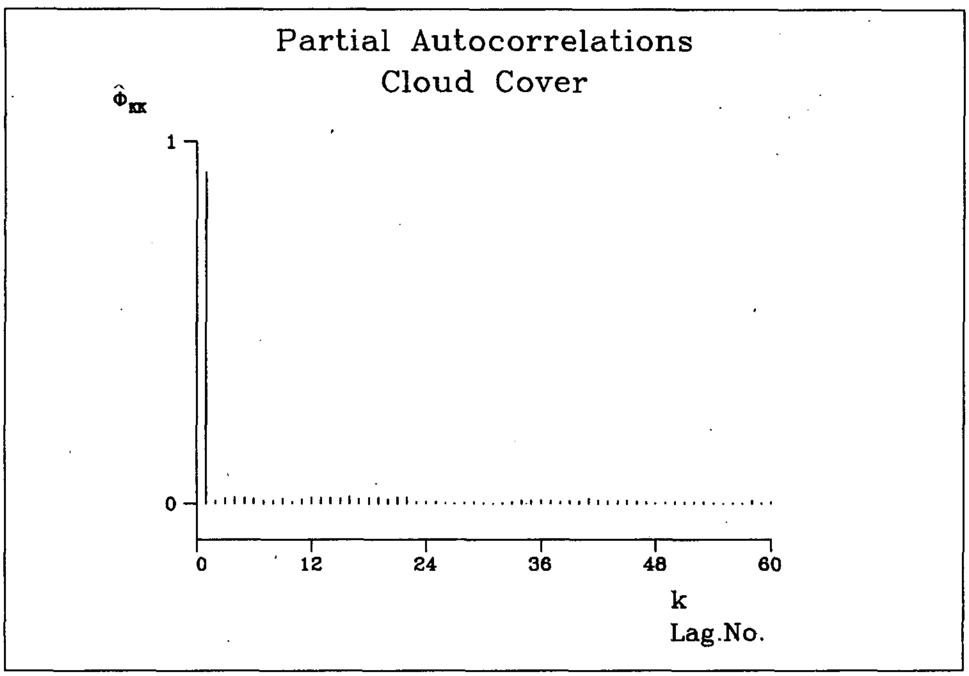

FIG. 4. Estimated partial autocorrelation function.

Considering only a homogeneous Markov process, we note that the one-hour transition probabilities are assumed to be independent of time and defined as (e.g., Cox and Miller, 1980)

$$p_{jk} = \text{prob}\{X_{t+1} = k | X_t = j\}, \tag{1}$$

from which the matrix of transition probabilities is found as

$$P = \{p_{jk}\}. \tag{2}$$

Ignoring the initial state of the process, the conditional likelihood function becomes

$$L(P) = \prod_j \prod_k p_{jk}{}^{n_{jk}} \tag{3}$$

where n_{jk} is the observed number of jumps from j to k in the series observations. The ML (Maximum Likelihood) estimate for p_{jk} is (Anderson and Goodman, 1957)

$$\hat{p}_{jk} = n_{jk}/n_{j\cdot} \tag{4}$$

where $n_{j\cdot} = \sum_k n_{jk}$.

Previously, we mentioned the marked annual variation of the cloud cover, which implies that the transition probabilities are dependent on the season. As a compromise between a desirable amount of data and the need for estimates which describe the annual variation, we have chosen to estimate a model for every month. Thus the ML estimate of the matrix of transition probabilities is found for every month in the year; n_{jk}, $n_{j\cdot}$ and $\hat{p}_{jk}$ for January are shown in Table 1.

There is an obvious dependence between two successive observations. The probability of observing the same or a neighbouring cloud cover in successive observations is high, thus the transition probabilities tend to be largest in the diagonal. The larger the transition, the smaller is the corresponding probability.

A simulation model in discrete time can be directly formulated from the monthly estimated matrices of transition probabilities. One disadvantage of the model in discrete time is the large number of parameters, namely 90 for each month. In addition it is not obvious how the annual variation of the estimates should be smoothed in order to obtain fewer parameters.

6. Continuous time Markov model

a. Model and assumptions

The number of parameters required for a model in continuous time is considerably reduced as compared with the discrete time model. In a short time interval the only possible changes in cloud cover are transitions to neighboring cloud cover states. The

TABLE 1. Observed jump frequencies (n_{jk}) and the corresponding estimated one-hour transition probabilities ($\hat{p}_{jk}$). January.

	k										
j	0	1	2	3	4	5	6	7	8	9	Total
0	747 0.835	86 0.096	17 0.019	15 0.017	5 0.006	5 0.006	5 0.006	4 0.004	2 0.002	9 0.010	895
1	95 0.156	346 0.567	71 0.116	39 0.064	18 0.030	13 0.021	12 0.020	10 0.016	3 0.005	3 0.005	610
2	24 0.067	85 0.238	105 0.294	71 0.199	20 0.056	17 0.048	15 0.042	17 0.048	1 0.003	2 0.006	357
3	9 0.024	41 0.108	71 0.187	97 0.256	51 0.135	43 0.113	38 0.100	22 0.058	6 0.016	1 0.003	379
4	4 0.013	13 0.041	37 0.117	47 0.149	71 0.225	54 0.171	52 0.165	33 0.104	4 0.013	1 0.003	316
5	4 0.010	15 0.039	23 0.060	37 0.096	48 0.124	76 0.197	97 0.251	70 0.181	13 0.034	3 0.008	386
6	3 0.005	8 0.012	14 0.022	31 0.048	45 0.069	90 0.138	222 0.342	187 0.288	45 0.069	5 0.008	650
7	3 0.002	5 0.003	16 0.009	23 0.014	41 0.024	62 0.037	138 0.082	1001 0.594	386 0.229	11 0.007	1686
8	3 0.001	6 0.001	3 0.001	15 0.003	16 0.003	24 0.005	63 0.013	327 0.066	4295 0.872	173 0.035	4925
9	4 0.004	2 0.002	0 0.0	3 0.003	3 0.003	2 0.002	8 0.009	14 0.015	171 0.182	734 0.780	941

cloud cover does not change momentarily from, e.g. state 2 to state 5. Therefore, basically, the only parameters necessary for the continuous time model are parameters describing transitions between neighboring cloud cover states. There are, however, two exceptions to this principle. Considering the matrix of transition probabilities in Table 1 of Section 5 we observe that the transition probability from state 0 to state 9 is higher than the transition probability from state 0 to, e.g., state 8. Likewise, the probability of transition from state 9 to state 0 is higher than the probability of transition from state 9 to state 1. This can be explained physically by the fact that foggy weather (state 9) can replace a clear sky (state 0) after sunset due to radiative cooling of the lower air stratum. Transitions from state 9 to state 0 are possible due to radiative heating after sunrise.

Introducing

$$p_{i,j}(\Delta t) = \operatorname{prob}\{X_t = j | X_{t-\Delta t} = i\} \tag{5}$$

the observations above lead to a model based on the following assumptions for $\Delta t \to 0$:

$$\left.\begin{aligned} p_{i,j}(\Delta t) &= o(\Delta t), \quad |i-j| \geq 2 \\ p_{i,i}(\Delta t) &= 1 - q_i\Delta t + o(\Delta t), \\ p_{i,i-1}(\Delta t) &= w_i q_i \Delta t + o(\Delta t), \\ p_{i,i+1}(\Delta t) &= (1 - w_i) q_i \Delta t + o(\Delta t), \\ i &\in \{0, 1, \ldots, 9\}, \end{aligned}\right\} \tag{6}$$

where q_i is the transition rate (intensity for leaving, or reciprocal mean waiting time), and w_i is the frequency of transitions to a lower cloud cover category. The notation includes transitions from 0 to 9 and the reverse, if we define state (-1) = state (9) and state (10) = state (0).

Due to the described assumptions the model is limited to be a first-order homogeneous Markov process in continuous time, and the assumptions lead directly to Kolmogorov's differential equation (e.g., Cox and Miller, 1980)

$$\frac{dP(t)}{dt} = P(t)Q; \quad P(0) = I \tag{7}$$

where $P(t) = \{p_{i,j}(t)\}$, and the matrix of transition rates

$$Q = \begin{bmatrix} -q_0 & (1-w_0)q_0 & 0 & \cdot & 0 & w_0q_0 \\ w_1q_1 & -q_1 & (1-w_1)q_1 & \cdot\ \cdot & 0 & 0 \\ 0\ \cdot & \cdot\ w_iq_i & \cdot & -q_i & (1-w_i)q_i & \cdot\ 0 \\ (1-w_9)q_9 & 0 & 0 & & w_9q_9 & -q_9 \end{bmatrix}. \tag{8}$$

Given the matrix of transition rates, the matrix of transition probabilities is found by solving Kolmogorov's differential equation, that is,

$$P(t) = e^{Qt}P(0) = e^{Qt}. \tag{9}$$

Compared with the discrete time model we note that the number of parameters is reduced from 90 to 20.

b. Simple estimates

Based on the monthly estimated transition probabilities simple estimates can be suggested. An intuitive estimate of the transition rate q_i can be based on the transition probability p_{ii}, as

$$\tilde{q}_i = -\ln \hat{p}_{ii}. \tag{10}$$

This estimate does not take into account that the cloud cover may change from a given state and back again within one hour, that is, in the sampling interval. Thus, evidently, the simple estimate will underestimate q_i, but for $q_i \ll 1\ \mathrm{h}^{-1}$ the error will be negligible.

Simple estimates for the frequency of transition to lower cloud cover are found in the same way as

$$\left.\begin{aligned} \tilde{w}_i &= \sum_{j=0}^{i-1} \hat{p}_{ij}/(1-\hat{p}_{ii}), \quad i = 1, 2, \ldots, 8, \\ \tilde{w}_0 &= (\hat{p}_{08} + \hat{p}_{09})/(1-\hat{p}_{00}), \\ \tilde{w}_9 &= (1 - \hat{p}_{90} - \hat{p}_{91})/(1-\hat{p}_{99}). \end{aligned}\right\} \tag{11}$$

For the same reason as above these estimates do not take into account all the possible combinations of transitions within one hour. The simple estimates are depicted in Figs. 5 and 6.

c. Maximum likelihood (ML) estimates

Since the observations of the cloud cover are by the hour, the matrix of one-hour transition probabilities becomes (t = 1)

$$P = P(1) = e^{Q}. \tag{12}$$

The logarithm of the conditional likelihood function is

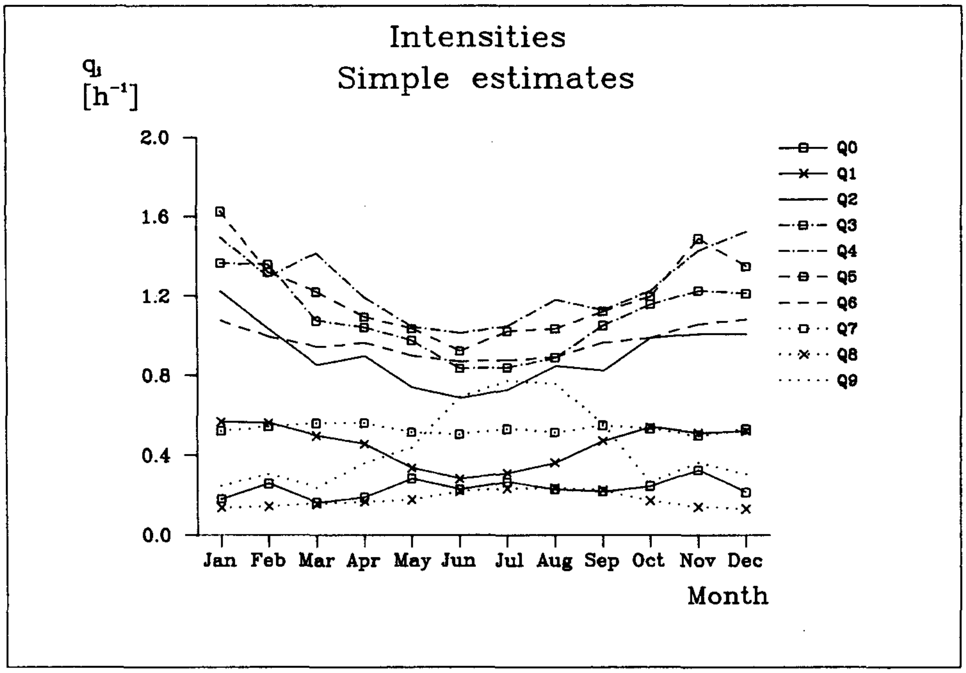

FIG. 5. Simple estimates of transition rates.

$$\log L(P) = \sum_i \sum_j n_{ij} \log p_{ij} \tag{13}$$

where n_{ij} is the total number of jumps from i to j in the series of observations.

The ML-estimate is the set

$$\hat{\theta} = (\hat{q}_0, \ldots, \hat{q}_9, \hat{w}_0, \ldots, \hat{w}_9) \tag{14}$$

which maximizes the likelihood function. From the general likelihood theory we have in regular cases

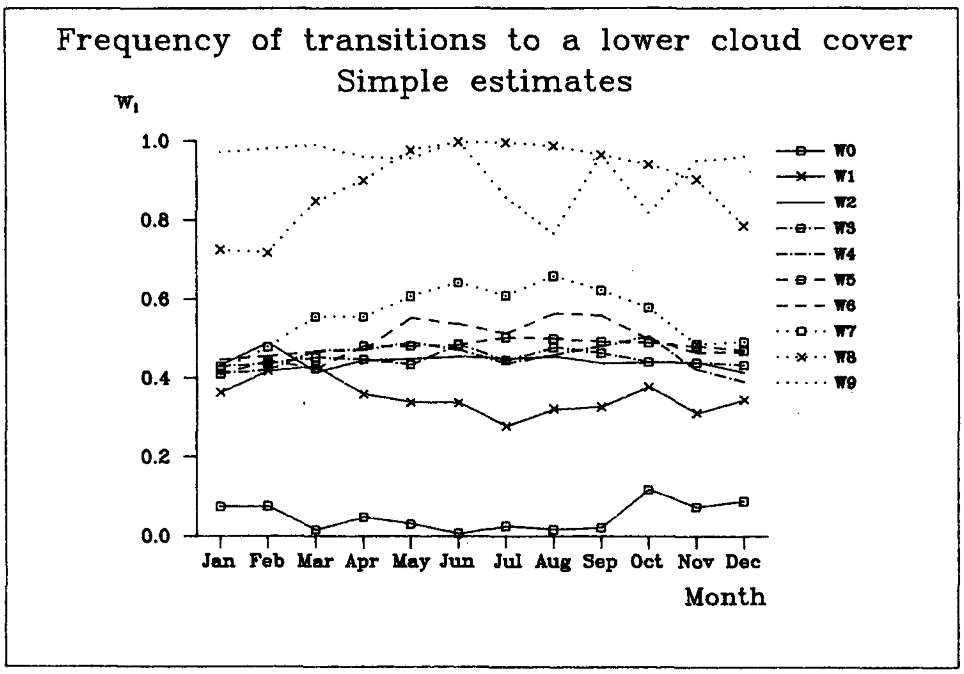

FIG. 6. Simple estimates of the frequency of transitions to a lower cloud cover category.

that the ML-estimator $\hat{\theta}$ is asymptotically normally distributed with mean θ and variance-covariance matrix

$$D = H^{-1}, \tag{15}$$

where the matrix H is given by

$$\{h_{lk}\} = -E\left[\frac{\partial^2}{\partial\theta_l\partial\theta_k}\log L(P)\right]. \tag{16}$$

An estimate of D is obtained by equating the observed value with its expectation and applying

$$\{h_{lk}\} \simeq -\left(\frac{\partial^2}{\partial\theta_l\partial\theta_k}\log L(P)\right)_{|\theta=\hat{\theta}}. \tag{17}$$

Thus, approximate confidence limits can be based on the normal distribution.

Since it has not been possible to determine an explicit expression for the ML-estimator $\hat{\theta}$, numerical methods have been used. Important problems in this connection are the computation of the exponential of the Q-matrix and the optimization of the likelihood function. The methods used for solving these problems are briefly described in Section 7.

The ML-estimates are shown in Figs. 7 and 8. As expected, a marked annual variation is observed. The estimates may be used to gain insight in the transitions between categories of cloud cover. For example it is seen that overcast sky (8) and clear sky (0) are the most stable states. The middle categories are seen to be the most unstable.

Comparing the ML-estimates of q_i with the simple estimates we note that the latter have been lifted and mostly for the largest values of q_i—as expected. The ML-estimates of w_i do not deviate much from the simple estimates.

The estimate of the parameters in the continuous time model may be utilized to calculate estimates of transition probabilities $P(t)$ corresponding to an arbitrary time-span t by means of (9). Particularly we can use the estimate of $(q_0, \ldots, q_9)$ and $(w_0, \ldots, w_9)$ to derive a smoothed estimate of the one-hour transition probabilities p_{ij}. Comparing these smoothed estimates with the observed frequencies of one-hour transitions we obtain a crude check on the fit of the model.

In Table 2 we have shown the smoothed estimate of the one-hour transition probability matrix for January, derived from the continuous time model by means of (9) and the ML-estimate of the parameters.

A comparison with Table 1, which contains the observed frequencies of jumps and the estimates by (4) of the one-hour transition probabilities, shows a good agreement. The relatively largest deviations are found far away from the diagonal. A major part of the deviations can possibly be ascribed to judgmental errors by the subjective observation of cloud cover. Another part of the deviations expresses the sampling fluctuation. Finally, some model deviation is quite natural since we restrict the attention to homogeneous Markov models only.

A similar comparison for the rest of the months of the year shows a still better agreement—especially for the months in the summer period. Since the judgmental errors are largest in the dark, and the

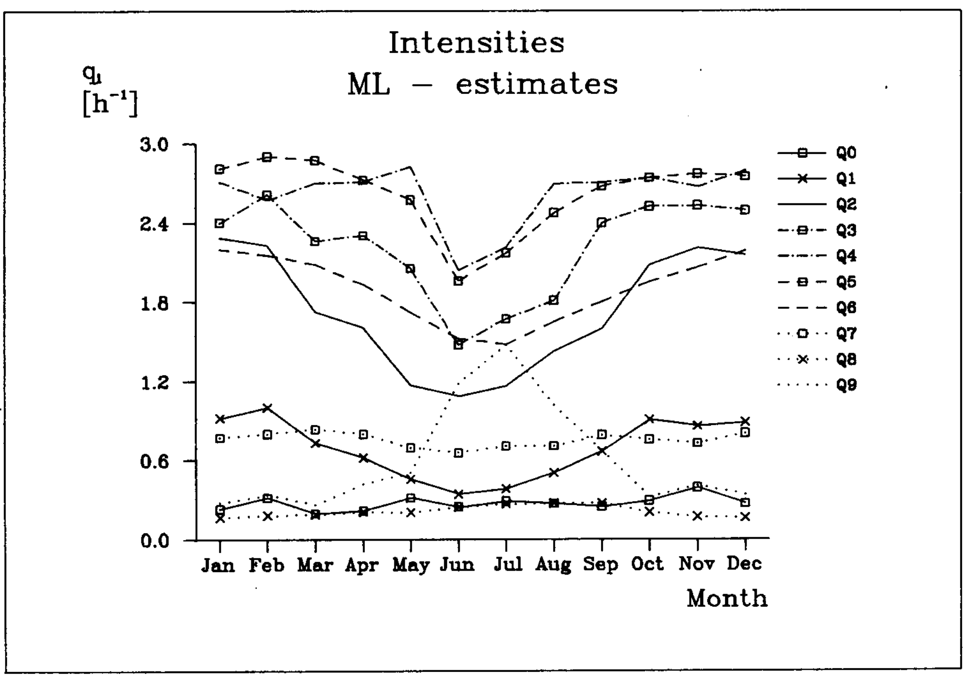

FIG. 7. ML-estimates of transition rates.

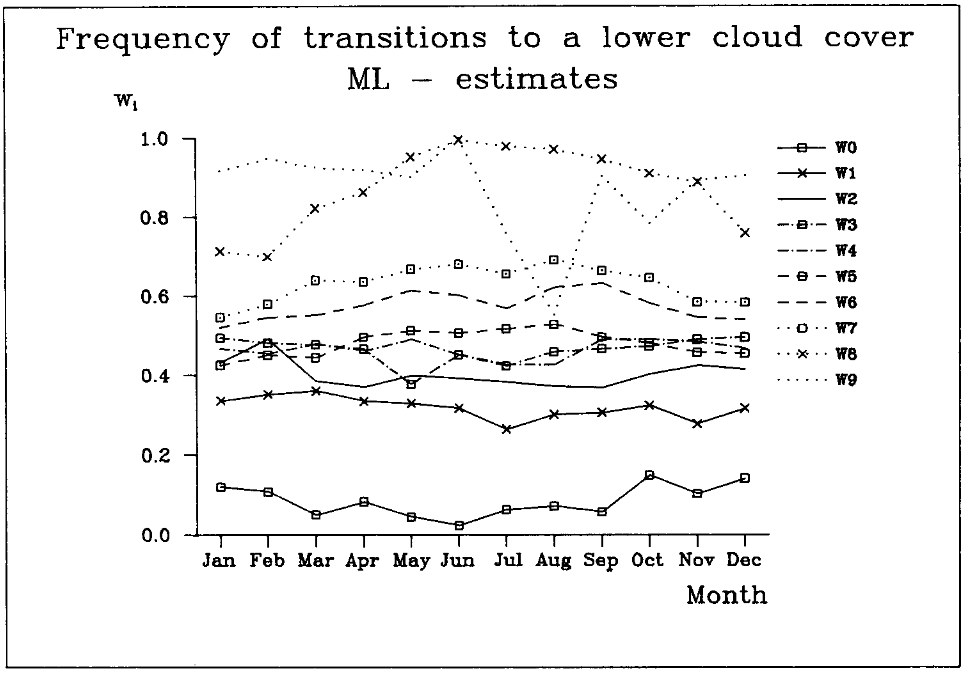

FIG. 8. ML-estimates of the frequency of transitions to lower cloud cover.

daily period of darkness is relatively large in January, this confirms our belief that a significant part of the deviations between the matrices in Tables 1 and 2 can be ascribed to judgmental errors.

Confidence bands of 90% for certain estimates are shown in Figs. 9 and 10. The uncertainty of the remaining transition rates is comparable with the uncertainty of $\hat{q}_1$. The coefficients of variation for $\hat{q}_0$, $\hat{q}_1$, $\cdots$ $\hat{q}_8$ are of the magnitude 3–8%. Due to the fact that fog is seldom present in the summer season, the uncertainty of $\hat{q}_9$ is higher in these periods. The uncertainty of the other estimates of frequency of transitions is approximately as indicated by the uncertainty of $\hat{w}_1$.

Considering the uncertainties, it would be reasonable to smooth the annual variation for the estimates in certain applications.

7. Numerical methods

The problems of optimization of the likelihood function and calculation of the exponential of the Q-matrix will be briefly considered in this section.

The IMSL-routine ZXMIN (IMSL, 1980) for unconstrained optimization was used. The routine is based on the David–Fletcher–Powell variant of the quasi-Newton method, as described by Luenberger (1973). To account for the bounded parameter space,

$$q_i \in [0, \infty],$$

$$w_i \in [0, 1],$$

quadratic penalty functions were introduced, as suggested by Luenberger.

The routine ZXMIN only finds a local maximum, but several facts contribute to our belief that the local

TABLE 2. One-hour transition probabilities derived from the continuous time model. January.

	k									
j	0	1	2	3	4	5	6	7	8	9
0	0.8142	0.1262	0.0246	0.0079	0.0019	0.0004	0.0001	0.0001	0.0029	0.0218
1	0.1912	0.5070	0.1765	0.0842	0.0268	0.0082	0.0025	0.0005	0.0003	0.0028
2	0.0601	0.2846	0.2648	0.2220	0.1039	0.0425	0.0166	0.0044	0.0004	0.0006
3	0.0176	0.1241	0.2030	0.2820	0.1929	0.1068	0.0536	0.0182	0.0017	0.0002
4	0.0043	0.0412	0.0988	0.2007	0.2371	0.2020	0.1409	0.0667	0.0081	0.0001
5	0.0009	0.0104	0.0336	0.0922	0.1676	0.2409	0.2484	0.1767	0.0290	0.0004
6	0.0002	0.0022	0.0093	0.0328	0.0829	0.1761	0.2773	0.3378	0.0800	0.0014
7	0.0001	0.0002	0.0010	0.0045	0.0157	0.0503	0.1356	0.5509	0.2358	0.0059
8	0.0004	0.0000	0.0000	0.0001	0.0007	0.0028	0.0110	0.0805	0.8655	0.0389
9	0.0181	0.0015	0.0002	0.0001	0.0001	0.0002	0.0010	0.0106	0.2058	0.7624

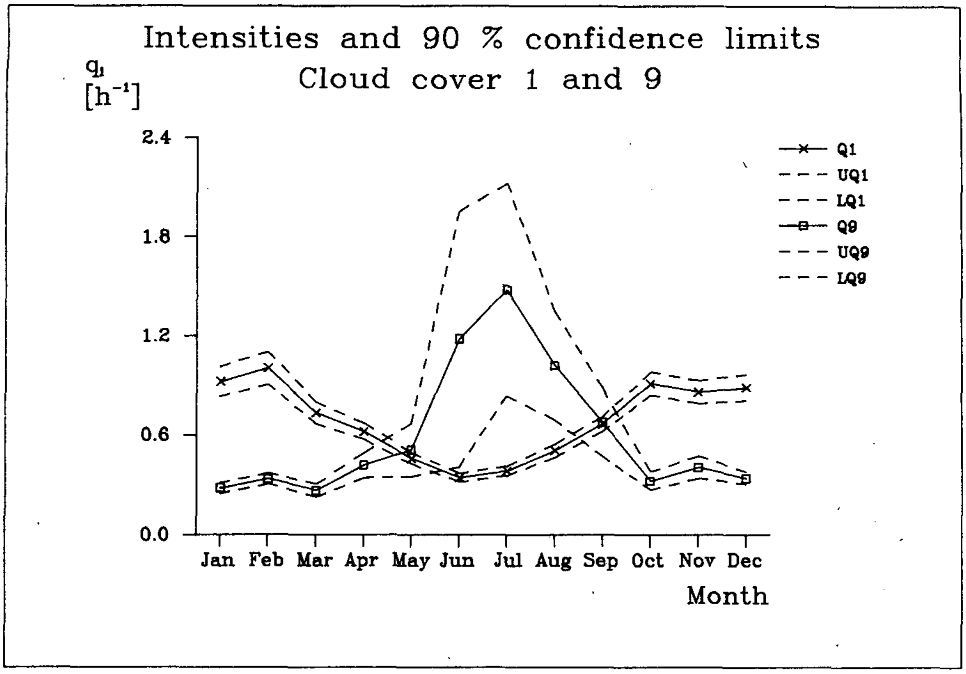

FIG. 9. 90% confidence band for $\hat{q}_1$ and $\hat{q}_9$.

maximum corresponds to the global maximum. First of all, we observed the expected connection between the simple estimates and the ML estimates. Further, it is noted that the same maximum was found independent of quite different starting values for the iterative search. Singer and Spilerman (1975, 1976) have discussed the uniqueness of the solution to $P = \exp(Q)$. It may be verified that if Q is a tridiagonal matrix, the solution is unique. In cases where Q has the structure given by (8) there may be more than

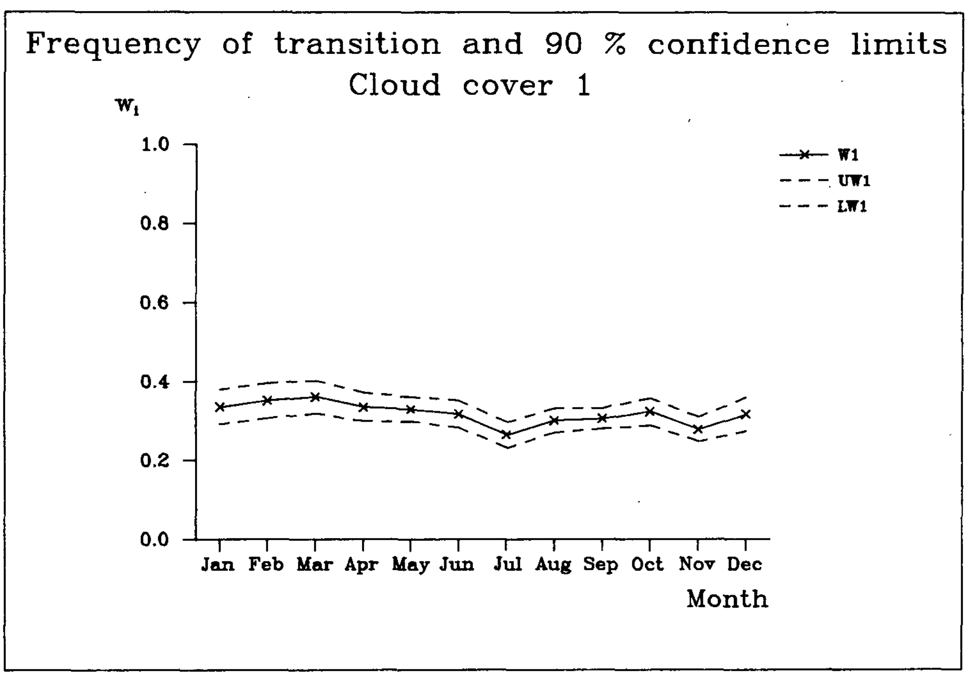

FIG. 10. 90% confidence band for $\hat{w}_1$.

one solution when all w_i are zero. In the present situation the data indicate that, e.g., w_9 is close to 1 and hence that the solution to $P = \exp(Q)$ is unique. Finally, during the search the routine never found areas where the likelihood function was not at least pseudoconcave.

The iterative search for the maximum implies repeated evaluation of $\exp(Q)$. Moler and van Loan (1978) have given a general discussion of the problems associated with the computation of the exponential of a matrix. In their paper, 19 different methods are described. The performance of these methods depends on the characteristic of the matrix. In our study it turned out that a combination of two techniques, namely a so-called Padé approximation and a method called 'scaling and squaring,' gave numerically stable results. In other situations alternative methods can be preferable.

8. Summary and future work

Fifteen years of hourly observations of cloud cover have been analyzed in order to obtain models of the variations in cloud cover. A very clear annual variation of cloud cover was found.

The cloud cover is measured in discrete time and in the categories 0 through 9. The categorized observations and the structure of the hourly correlations make it attractive to consider Markov processes. As a natural starting point, homogeneous processes have been considered. In order to describe the annual variation, a model is set up for every month.

Although the data are available in discrete time only, we consider models in both discrete and continuous time. Not only is the cloud cover variation a continuous time phenomena, but, furthermore, the continuous time model requires considerably less parameters than the corresponding discrete time model because, generally, in continuous time transitions only to neighboring cloud cover categories are possible.

The estimation of the parameters of the different models has been described. The matrix of transition rates belonging to the embedded Markov process in continuous time is estimated by a simple method and by the maximum likelihood method. The ML-estimate was found by numerical methods using an iterative search for the maximum of the likelihood function. Confidence bands have been obtained as an indication of the uncertainty associated with the parameter estimates. We obtained monthly parameter estimates, and for certain purposes it would be natural to interpolate or filter the monthly values in order to get a smoothed annual description. The obtained confidence bands indicate that this procedure could be very reasonable.

A possible drawback of the homogeneous model considered is that the diurnal variation is neglected. The diurnal variation could be described by introducing inhomogeneous Markov processes. Thus, the matrix of transition rates and the transition probabilities could depend on the time of the diurnal cycle. In this situation Kolmogorov's differential equation becomes (Cox and Miller, 1980, p. 181)

$$\frac{\partial P(t, u)}{\partial u} = P(t, u)Q(u) \tag{18}$$

where u denotes the time of the day. The structure of $Q(u)$ is as in the homogeneous case. An approximate solution is obtained by assuming Q to be constant in periods of length T. It is natural to choose T as a number of sampling periods ($T = n$ hours). Thus, the number of required parameters will be $(20 \times 24/n)$ for each month. The number of parameters in continuous time is again less than the number of parameters in discrete time which is $(90 \times 24/n)$ for each month. An additional reduction of the numbers of required parameters can be obtained by modeling the variation of $Q(u)$ by simple functions, for example by some terms of a Fourier expansion. Furthermore some of the parameters could be fixed after testing the influence of the diurnal variation. Work on these problems is in progress.

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