Proceedings of the
American Control Conference
Seattle, Washington • June 1995

Optimal Input Design for Fault Detection and Diagnosis

P. SADEGH*, H. MADSEN* and J. HOLST†

* Inst. of Mathematical Modeling, Technical University of Denmark, DK-2800 Lyngby, Denmark.

† Dept. of Mathematical Statistics, Lund Institute of Technology, S-22100 LUND, Sweden.

Abstract

In the paper, the design of optimal input signals for detection and diagnosis in a stochastic dynamical system is investigated. The design is based on maximization of Kullback measure between the model under fault and the model under normal operation conditions. It is established that the optimal input design for change detection when the magnitude of change is small is equivalent to optimal input design for parameter estimation.

1. Kullback measure and input design

Definition 1.1 *The Kullback-Leiber information for discriminating between two models $\mathcal{M}_0$ and $\mathcal{M}_1$ is defined by*

$$K(\mathcal{M}_1, \mathcal{M}_0) = \int \log \frac{p_1(y)}{p_0(y)} p_1(y) dy$$

where $p_0(y)$ and $p_1(y)$ are the probability densities of the data under $\mathcal{M}_0$ and $\mathcal{M}_1$ respectively.

Theorem 1.1 *Assume that the two SISO models:*

$$\begin{aligned} \mathcal{M}_0 &: y(k\Delta T) = G_{1,0}(\delta)u(k\Delta T) + G_{2,0}(\delta)e(k\Delta T) \\ \mathcal{M}_1 &: y(k\Delta T) = G_{1,1}(\delta)u(k\Delta T) + G_{2,1}(\delta)e(k\Delta T) \end{aligned}$$

are to be tested against each other where the δ-operator is defined by (cf [1])

$$\delta = \frac{q-1}{\Delta T}$$

q is the forward shift operator and ΔT is the sampling time. Also assume that

- *$\{e(k\Delta T)\}$ is a sequence of uncorrelated zero mean Gaussian random variables.*
- *The input signal admits a spectral representation and is power restricted.*
- *The length of the experiment is large.*
- *There is no feed-back in the system.*

Then maximizing the Kullback-Leiber measure $K(\mathcal{M}_1, \mathcal{M}_0)$ with respect to the power restricted input is equivalent to the optimization problem

$$\begin{aligned} &\max \int_0^{\frac{\pi}{\Delta T}} \left|\frac{\Delta G_1'(e^{j\omega\Delta T})}{G_{2,0}'(e^{j\omega\Delta T})}\right|^2 d\xi(\omega) \\ &\int_0^{\frac{\pi}{\Delta T}} d\xi(\omega) = 1 \end{aligned} \tag{1}$$

where for each transfer function, we denote $G(\delta) = G'(q)$. $\xi(\omega)$ is the power distribution of the input defined over the range of frequencies $[0, \frac{\pi}{\Delta T}]$. The solution is given by

$$\xi(\omega) = \begin{cases} 0 & \omega < \omega^* \\ 1 & \omega \geq \omega^* \end{cases} \tag{2}$$

where

$$\omega^* = \arg\max_{\omega} \left|\frac{\Delta G_1(\frac{e^{j\omega\Delta T}-1}{\Delta T})}{G_{2,0}(\frac{e^{j\omega\Delta T}-1}{\Delta T})}\right| \tag{3}$$

and

$$\Delta G_1(\delta) = G_{1,1}(\delta) - G_{1,0}(\delta)$$

PROOF: *The proof is based on direct computation of the Kullback measure. For details see [2].*

Remark 1.1 If we denote the parameter in $G_{1,0}$ by θ, then for small changes $\Delta\theta$, the criterion given by Eq(1) will be approximately proportional to

$$\begin{aligned} &\Delta\theta^T \{\int_0^{\frac{\pi}{\Delta T}} G_{2,0}'^{-1}(e^{j\omega\Delta T}) \times \\ &Re\{[\frac{\partial G_{1,0}'(e^{j\omega\Delta T})}{\partial\theta}][\frac{\partial G_{1,0}'^{-1}(e^{-j\omega\Delta T})}{\partial\theta}]^T\} \times \\ &G_{2,0}'^{-1}(e^{-j\omega\Delta T}) d\xi(\omega)\} \Delta\theta \end{aligned} \tag{4}$$

Letting $\Delta T \to 0$, we obtain the optimization criterion for change detection in continuous-time models. It will be given by

$$\begin{aligned}&\Delta\theta^T\{\int_0^\infty G_{2,0}^{-1}(j\omega)\times\\ &Re\{[\frac{\partial G_{1,0}(j\omega)}{\partial\theta}][\frac{\partial G_{1,0}(-j\omega)}{\partial\theta}]^T\}\times\\ &G_{2,0}^{-1}(-j\omega)d\xi(\omega)\}\Delta\theta\end{aligned} \tag{5}$$

For easy reference, we denote Eq(4) or Eq(5) by $\Delta\theta^T M_\theta \Delta\theta$.

2. Connection to Input Design for Parameter Estimation

It can be shown [3] that under the assumptions of Theorem 1.1, the information matrix corresponding to $\mathcal{M}_0$ can be written as $M_u + M_c$ where M_c is not input dependent and M_u is proportional to M_θ.
The information matrix asymptotically gives the covariance of any unbiased efficient estimator of θ. Hence, maximizing a scalar function of the information matrix with respect to design variables is the topic of the statistical experiment design for precise estimation of parameters.
Now assume that only the changes in a subset of θ is of interest and should be monitored. We have the following theorem

Theorem 2.1 *Assume that the parameter vector θ is partitioned into $\theta = (\theta_1^T, \theta_2^T)^T$, and M_θ is partitioned*

$$M_\theta = \left[\begin{array}{cc} M_{11} & M_{12} \\ M_{12}^T & M_{22}\end{array}\right] \tag{6}$$

accordingly. It is desired to test the small changes in θ_1. The minimum (worst) value of $\Delta\theta^T M_\theta \Delta\theta$ with respect to $\Delta\theta_2$ is assumed at $\Delta\theta_2 = -M_{22}^{-1}M_{12}^T\Delta\theta_1$. This minimum value is

$$\Delta\theta_1^T(M_{11} - M_{12}M_{22}^{-1}M_{12}^T)\Delta\theta_1 \tag{7}$$

PROOF: *The proof is straightforward and based on minimizing the Kullback information with respect to $\Delta\theta_2$. For easy reference, we denote Eq(7) by $\Delta\theta_1^T M_\theta' \Delta\theta_1$.*

Optimal input design is thus reduced to maximization of $\Delta\theta M_\theta \Delta\theta$ or $\Delta\theta_1 M_\theta' \Delta\theta_1$ subject to constraints. It requires *a priori* knowledge about the change $\Delta\theta$ or $\Delta\theta_1$. In absence of such prior information, one may for example maximize the determinant of M_θ or M_θ'. These choices of optimization criterion are respectively known as D and D_s optimality in the statistical experiment design literature [4].

3. Optimal Inputs for Diagnosis

Assume that the parameter of the model θ is a function of some physical parameter denoted by β, i.e. $\theta = F(\beta)$. It is of interest to monitor changes in β. Replacing $\Delta\theta \approx \frac{\partial F}{\partial\beta}\Delta\beta$, the optimization criterion is approximately given by $\Delta\beta^T M_\beta \Delta\beta$ where

$$M_\beta = (\frac{\partial F}{\partial\beta})^T M_\theta (\frac{\partial F}{\partial\beta})$$

Now, partition $\beta = (\beta_1^T, \beta_2^T)^T$ and assume that only the subset β_1 is of interest. Partition M_β as in Eq(6) and obtain the criterion $\Delta\beta_1^T M_\beta' \Delta\beta_1$ defined similar to Eq(7), cf [2].

4. Conclusion

In the paper, we have discussed the question of design of optimal input signals for detection and diagnosis. Our suggested design of inputs is based on full knowledge of the process in the normal operating conditions. However, no information about the possible faults is assumed given except for that they are *small.* The equivalence between optimal input design for fault detection and parameter estimation is established.

References

[1] R. H. Middleton and G. C. Goodwin. *Digital Control and Estimation.* Prentice Hall Information and System Science Series. Prentice Hall, 1990.

[2] P. Sadegh, H. Madsen, and J. Holst. Optimal input design for fault detection and diagnosis. Technical report, Institute of mathematical Modeling, Lyngby, Denmark, 1995.

[3] G. C. Goodwin and R. L. Payne. *Dynamic System Identification: Experiment Design and Data Analysis.* Academic Press, 1977.

[4] V. V. Fedorov. *Theory of Optimal Experiments.* Academic Press, 1972.